REPRESENTATIVITY AND NON-CONTRACTIBLE SEPARATING CYCLES OF EMBEDDINGS ON THE TRIPLE TORUS

MASTER OF SCIENCE IN MATHEMATICAL SCIENCES THESIS

PRECIOUS JUWAWO

UNIVERSITY OF MALAWI

MAY, 2024

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BY

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BEd. Science - University of Malawi

Submitted to the Department of Mathematical Sciences, School of Natural and Applied Sciences in (partial) fulfillment of the requirements for the award of the degree of Master of Science in Mathematical Sciences.

UNIVERSITY OF MALAWI

MAY, 2024

DECLARATION

I, **Precious Juwawo**, declare that this project is my own original work and that it has not been presented and will not be presented to any other university for a similar or any other degree award. Where somebody's work or results have been used, proper acknowledgements have been made.

Signed

______________________________ **Date:** Friday 7th June, 2024.

Precious Juwawo

CERTIFICATE OF APPROVAL

The undersigned certify that this thesis represents the student's own work and effort and has been submitted with their approval.

Signed

________________________________ **Date:** Friday 7th June, 2024.

Associate Professor Patrick Ali (Supervisor)

ACKNOWLEDGEMENTS

First and foremost, I owe my thanks to God for bringing me this far. He has been faithful throughout my academic pursuits.

Secondly, my acknowledgement goes to my supervisor, Associate Professor Patrick Ali who is also my role model in my career as a Mathematician. I would like to thank him for suggesting this interesting topic of research and for showing such high skill of mastery of content in the topic, portrayed by comments and directions I received from him. Without him, this project would not have been successful.

I also wish to extend my gratitude to the Mathematical Sciences Department of the University of Malawi for their support ranging from offering me the NORPART scholarship to giving me a job as a part-time lecturer and tutor.

The following lecturers are acknowledged for teaching our classes in a unique manner during course work at the University of Stavanger in Norway; Paul Demedeiros, Tyson Ritter, and Alexander. Special thanks to Paul Demedeeiros for teaching Manifolds and Surface Topology in a simplified and understandable manner, through which I got background knowledge in handling this research topic.

Finally, I would like to acknowledge my church mates, members of Living Waters Church Matawale for their spiritual support, and my classmates, Beatrice, Augustine, Sharif, Akuonani and Omega for their academic support.

DEDICATION

I dedicate this project to my wife Fortunate E. Kayuni who has been standing with me throughout my pursuit of MSc degree.

ABSTRACT

We consider Ellingman's and Zhao's method of proving that every 4 representative graph embedding on the double torus contains a Non-Contractible Separating Cycle (NSC). They proved this main result by considering critical embeddings; which are embeddings that are very close to having NSCs. We adopt the method in proving an extension of the same theorem to a surface of one genus higher; the triple torus. The method works efficiently in proving our main result that every 4 representative embedding on the triple torus contains two NSCs which separates the triple torus into 3 connected components, namely punctured tori, two of them with one boundary circle and one with two boundary circles. Our results are obtained by employing equivalence of embeddings and homeomorphism of surfaces to Ellingman and Zhao's method.

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NSC	Non Contractible Separating Cycle.

CHAPTER 1

INTRODUCTION

This chapter presents terminologies and some notations that will be used in subsequent chapters and sections of the thesis. We first present terminologies on graphs, surfaces, and embeddings of graphs on surfaces in the background. We also present objectives of the study, problem statement and significance of the study in this chapter.

1.1 Background

1.1.1 *Graphs*

A graph G is a pair of sets, $V(G)$ and $E(G)$, where $V(G)$ is nonempty and $E(G)$ is a set of 2-element subsets of $V(G)$. A directed graph G is an ordered pair $G = (V(G), A(G))$ consisting of a set $V = V(G)$ of vertices and a set $A = A(G)$ of arcs, together with an incidence function ψ_G that associates with each arc of G an ordered pair of vertices of G. The number of vertices, $n = |V(G)|$, is the order of the graph G. A walk in the graph $G = (V, E)$ is a finite sequence of the form $v_{i_0}, e_{j_1}, v_{i_1}, e_{j_2}, \ldots, e_{j_k}, v_{i_k}$, which consists of alternating vertices and edges of G.

The walk starts at a vertex. A walk is open if $v_{i_0} \neq v_{i_k}$. We refer the reader to [**11**, **4**] for further details.

The graph G is disconnected if it is the disjoint union of two other graphs and connected otherwise. A connected graph is said to be k-vertex connected if it has more than k vertices and remains connected whenever fewer than k vertices are removed. The vertex connectivity or just connectivity of a graph G is the largest k for which the graph is k-vertex-connected, we refer to [**4**].

Two edges with a common end are said to be adjacent. An isomorphism of graphs G and H is a 1-1 mapping ψ of $V(G)$ onto $V(H)$ such that adjacent pairs of vertices of G are mapped to adjacent vertices in H, and nonadjacent pairs of vertices have nonadjacent images. Two graphs G and H are said to be isomorphic if there is an isomorphism between G and H. If $G \cong H$ then G and H are isomorphic, see [**11**].

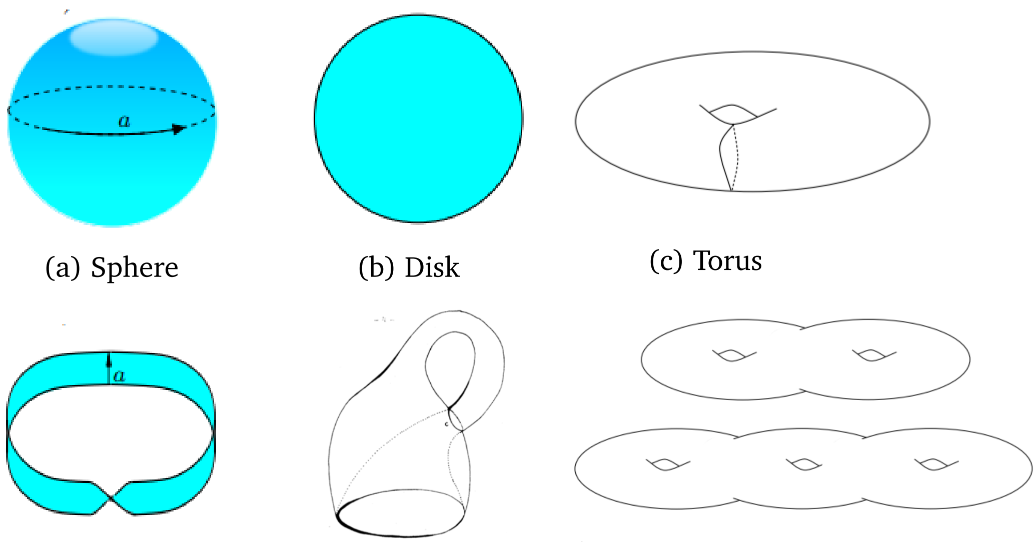
A graph is planar if it can be drawn in the plane in such a way that no edges intersect, except of course at a common end vertex. In other words planar graphs are embeddable on the plane. While non planar graphs cannot be embedded on the plane, they can be embeddable on surfaces other than the plane, see [**11**].

1.1.2 *Surfaces*

Graphs can be studied on the sphere (plane), or on other surfaces. A surface is a compact two-dimensional manifold, possibly with boundary. Equivalently, a surface is a compact topological space that is Hausdorff (any two distinct points have disjoint neighbourhoods) and such that every point has a neighbourhood homeomorphic to a plane or a closed half plane. If each point of a surface has a neighbourhood homeomorphic to the plane, such a surface is closed, otherwise it is called a punctured surface or a surface with boundary, see [**11**].

If Σ is a punctured surface, then a point p on the surface whose neighbourhoods are homeomorphic to the upper half plane is said to be a boundary point. The union of boundary points of Σ is a collection of circles. These circles are boundary components of Σ. The genus of a connected, orientable surface is an integer representing the maximum number of cuttings along non-intersecting closed simple curves without rendering the resultant manifold disconnected, we refer to [**6**, **4**].

Examples of surfaces include sphere, cylinder, disk, torus, double torus, Mobius strip, Klein bottle and the projective plane among others. Some surfaces are shown in figure 1.1.



(a) Sphere (b) Disk (c) Torus

(d) Mobius strip (e) Klein bottle (f) Double torus (top), Triple torus (bottom)

Figure 1.1: Some examples of surfaces, see [4].

A surface is non-orientable if some subset of it (with induced topology) is homeomorphic to the Mobius strip, otherwise it is orientable [6]. Table 1.1 shows a brief classification of common surfaces.

Table 1.1: Classification of surfaces, see [4].

Surface	Orientable?	Genus	No. of boundary components
Sphere	yes	0	0
Disk	yes	0	1
Torus	yes	1	0
Double Torus	yes	2	0
Triple Torus	yes	3	0
Mobius strip	no	1	0
Klein bottle	no	2	0

Given two topological surfaces Σ_a and Σ_b, the map $h : \Sigma_a \to \Sigma_b$ is a homeomorphism if h is bijective and smooth (both h and its inverse are smooth). For any two closed connected surfaces Σ_a and Σ_b, then such surfaces are homeomorphic if and only if they are both orientable or both non-orientable and they have the same genus. This implies that every surface is homeomorphic to itself, see [**4**].

A path on a surface Σ is a continuous map $P : [0,1] \to \Sigma$. Its two endpoints are $P(0)$ and $P(1)$. A loop is a path whose two end points are equal. A closed curve on Σ is a continuous map from the unit circle S^1 to Σ, also called a cycle, we refer to [**4**].

1.1.3 *Graphs on Surfaces*

There are two definitions of graph embeddings, one is topological and the other combinatorial. In the topological sense, let G be a graph and Σ a surface. Then the embedding of G onto Σ can be viewed as a continuous map $\Psi : G \to \Sigma$, see [**4**].

Combinatorically, an embedding Ψ of a graph G on a surface Σ is a crossing free drawing of G on Σ. It maps the vertices of G to distinct points of Σ and its edges to paths of Σ whose endpoints are images of their incident vertices. The face of an embedding Ψ is a connected component of the complement of the image of Ψ as a map. A cellularly embedded graph $G = (V(G), E(G)) \subset \Sigma$ is a graph drawn in a surface Σ such that its edges only intersect at their ends and each connected component of $\Sigma \backslash G$ is homeomorphic to a disk, we refer to [**6**].

The connected components of $\Sigma \backslash G$ when viewed as subsets of Σ, are called faces of G. Two embeddings $\Psi_a : G \to \Sigma_a$ and $\Psi_b : G \to \Sigma_b$ are strictly equivalent if there is a homeomorphism $h : \Sigma_a \to \Sigma_b$ such that $\Psi_a = h \circ \Psi_b$. For a cellular embedding, two cellular embeddings of a graph G are equivalent if and only if their faces have same boundaries, see [**13**].

Σ-facial walks (or simply Σ-faces) are closed walks in the graph which correspond to traversals of face boundaries of the topological embedding related to Σ. In that sense, two embeddings are equivalent if they have the same facial walks. Let $\mathcal{F}(\Sigma, G)$ be the set of Σ-facial walks. The number $\chi(\Sigma) = |V(G)| - |E(G)| + |\mathcal{F}(\Sigma, G)|$ is called the Euler characteristic of the embedding of G onto Σ, we refer to [**9**].

A circle in a surface is a simple closed curve and an arc is a simple non-closed curve including its end points. There are 3 types of such cycles; contractible, non-contractible and surface separating. A cycle on a surface is contractible if it can be continuously deformed to a point without leaving the surface, otherwise it is non-contractible. It is separating if cutting the surface along such a cycle splits the surface into two connected components, otherwise it is non-separating, see [**4**].

Two cycles on a surface are homotopic if there is a continuous deformation of one onto the other, that is, if there is a continuous function from the cylinder $S^1 \times [0, 1]$ to Σ such that each boundary of the cylinder is mapped to one of the loops. In that sense, a cycle is contractible if it is homotopic to a constant loop (one whose image is a single point), see [**2**].

Figure 1.2 shows such 3 types of cycles; contractible (right), non-contractible and separating (center) and non-separating (left).

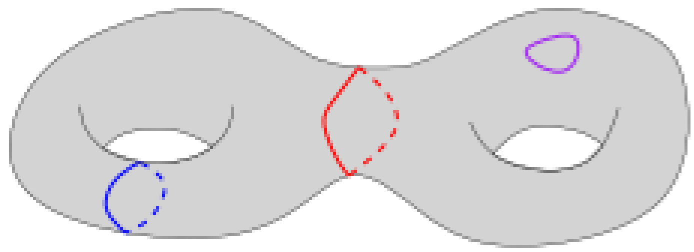


Figure 1.2: The three types of cycles on a double torus.

Suppose that C is a surface non-separating cycle of a Σ-embedded graph G. If C is Σ-two-sided, let $\bar{G}$ be the graph obtained from G by replacing C with two copies of C such that all edges on the left side of C are incident with one copy of C and all edges on the right side of C are incident with the other copy of C. We say that $\bar{G}$ is obtained from G by cutting (or Σ-cutting) along the cycle C, we refer to [**11**].

Representativity of an embedding, or face width of a graph embedding is the smallest number of points in which any non-contractible closed curve on a surface intersects the graph. Let G be a graph, Σ be a surface, and $\Psi : G \to \Sigma$ be an embedding of G in Σ. Representativity of an embedding can also be defined as a set $\rho(\Psi) = \min\{|\Gamma \cap G| : \Gamma$ is a Non-contractible simple closed curve on $\Sigma\}$. An embedding is critical if it is very close to having and NSC, see [**13**, **5**].

A single point is not considered to be an arc. The interior Q^o of an arc Q consists of an arc and its end points deleted. If a graph is embedded in a surface, each cycle of the graph is embedded as a circle, and each nontrivial (not a single vertex) path as an arc. A section of an arc or a circle on a surface is a subarc or single point contained in the arc or circle, we refer to [**5**].

T is used to denote the torus (a sphere with one handle). A graph G is toroidal if G embeds in T . Let Ψ be an embedding of $G = G(\Psi)$ in T. The closure of each connected component of $T \backslash G(\Psi)$ is called a face of Ψ (closed faces are mostly preferred to open ones). The face set of an embedding Ψ in T is denoted by $F(\Psi)$. If the graph is 2-connected and $\rho(\Psi) \geqslant 2$ then each face f is bounded by a cycle, called a facial cycle and is denoted by ∂f. ∂X denotes the boundary of a set $X \subset T$. Two vertices x and y are cofacial by a face f if $x, y \in \partial f$. Embeddings with $\rho(\Psi) \geqslant 4$ have all faces bounded by cycles in graphs, see [**13**].

If D is a closed disk of T with boundary contained in G, then ∂D (the boundary of D) is a cycle of G. Let $f \in F(\Psi)$ be a face of Ψ. The symmetric differences of ∂f and all the facial cycles incident to f is a union of cycles of G, and because $\rho(\Psi) \geqslant 4$, one of these cycles bounds a disk containing f together with all the faces incident to f. Such disk is named D_f which is also called the second disk of f. It was noted that the second disk D_f consists of the face f, all faces incident to f, and all faces that are surrounded by f, we refer to [**10**].

A segment of a path or a cycle in a graph is a subpath, which may consist of just a single vertex. If a, b are sections of an oriented arc or oriented circle Q then aQb denotes the part of Q from the last point of a to the first point of b, inclusive. Q^{-1} denotes Q traversed in the opposite direction. The number of components of a set S on a surface Σ is denoted $\parallel S \parallel$. For convinience, all topological sets are dealt with as closed sets, that is they include their boundaries. Contractible circles on T have natural clockwise orientation and non-contractible circles must be given an orientation, see [**5**].

1.2 Motivation

Several authors have written on cycles of graphs embedded on surfaces. Erickson wrote on an algorithm used to compute shortest essential cycle of graphs embedded on orientable combinatorial surfaces, see [**7**]. Ellingman N.& Zha. X, see [**5**], wrote on existence of NSCs in graph embeddings on surfaces. They proved that every 4-representative graph embedded on a double torus contains a NSC.

In this thesis, we wish to explore the existence of Non-contractible separating cycles in 4-representative graphs embedded on a triple torus. We seek to prove that every 4-representative graph embedded on a triple torus has two NSCs which together splits the surface into 3 connected components. Zha proved that every 6-representative embedding on a suitable orientable surface has an NSC, see [**13**]. We choose to work with 4-representative graph embeddings because it is possible to raise a 4-representative embedding to 6-representative by performing a series of augmentations. After raising the representativity from 4 to 6, at that point we are sure that a NSC exists, see [**5**].

We will consider Ellingma's and Zhao's method of proving that every 4-representative graph embedded on the double torus has NSC. By using equivalence of embeddings and homeomorphism of surfaces we extend their method to embeddings on a triple torus to prove our main theorem.

1.3 Statement of the problem

Graphs on surfaces is one of the interesting and emerging fields under graph theory. In recent decades, it has sparked research with many studies focusing on cycles of embedded graphs. Many authors have studied existence of Non Contractible Separating cycles of graphs embedded on suitable surfaces, which are defined to be those with genus at least 2.

When a graph is embedded on a surface, many questions may arise including the existence of cycles on the embedding and the nature of such cycles. For example, Ren studied cycle bases of embedded graphs. He argued that in graph embedding theory, a branch of topological graph theory, cycle operations have particular uses, see [**12**].

Some authors including Erickson & Hubard have written on finding shortest cycles of embedded graphs. These are cycles with minimum face widths of all other cycles of such embedded graphs. Under this adventure, the problem of finding shortest paths in non embedded graphs can be reflected but with a minor difference that such paths will be closed arcs on surfaces, see [**7**] and [**8**]

The field of manifolds which tries to perform calculus on abstract surfaces can also be well understood by studying cycles of embedded graphs. The concept of triangulation of surfaces is also well explained by studying cycles since triangulations are just subsets of 3-representative embeddings, see [**5**].

Now coming to non-contractible separating cycles on embedded graphs, authors have studied them on embedded graphs with the help of representativity of embeddings. Ellingman studied NSCs of graphs embedded on a torus, the suitable surface of genus 2. They proved that every 4- representative embedding on the double torus contains an NSC, see [**5**]. However they did not suggest a possible extension of their theorem to a surface of one genus higher, the triple torus. We find it an interesting adventure to study whether the line of argument in their proof could apply to a triple torus.

1.4 Research objectives

1.4.1 *Main objective*

The main objective of this study is to prove that every 4 representative graph embedding on the triple torus contains two non contractible separating cycles which separates the triple torus into 3 connected components.

1.4.2 *Specific objectives*

Specifically, the objectives of the study are;

(i) To analyse Ellingman's and Zhao's method of proving that every 4 representative embedding on the double torus contains a non-contractible separating cycle which separates the double torus into 2 connected components.

(ii) To apply Ellingman's and Zhao's method in proving that every 4 representative embedding on the triple torus contains 2 non-contractible separating cycles which separates the surface into 3 connected components

1.5 Significance of the study

The study of graphs on surfaces is of great significance in the field of graph theory. It gives a good link between the field of topology and that of graph theory, giving good insight into topological graph theory. On the other hand, it serves as a connection between the fields of combinatorics and graph theory which in turn gives an insight into Combinatorial graph theory. This makes it possible for both experts of topology and algebra to come together in understanding graph theory.

In topological graph theory, cycles of embedded graphs are of great interest. An understanding of cycles of graphs shall lead to an understanding of cycles on surfaces. This is significant since graph embeddings are topologically understood as maps into surfaces. Cycles of embedded graphs help in understanding representativity of embeddings since they create faces on surfaces and such faces play a pivotal role in defining reperesentativity of embeddings.

Studying graph embeddings is crucial and has some applications in the real world. The goals of graph embedding, such as minimizing edge crossings align very well with the objectives of mesh untangling. Meshes are a variety of graphs used to represent surfaces with a wide number of applications, particularly in simulation and modelling.

Non Contractibe separating cycles are the most interesting cycles of embedded graphs since they cannot be reduced to a single point on the surface and also because they separate surfaces leading to more interesting results. This study is significant since proving existence of NSCs in embeddings on the triple torus can lay a good foundation to the study of cycles of graphs embedded on surfaces of higher genus and on some other higher dimensional manifolds.

1.6 Limitations of the study

In literature most authors wrote on conditions for existence of NSCs in embedded graphs. Zha proved that every 6-representative embedding on a suitable orientable surface has an NSC, see [**13**]. We build our proof on this result by perfoming a series of augmentations on our 4-representative embeddings in order to raise the representativity to 6 at which point an NSC exists. These series of augmentations may tamper with the embedding if not well executed.

1.7 Organisation of the thesis

In chapter 2 we give a review of literature which focuses on conditions for existence of NSCs in graph embeddings and those that have proven existence of NSCs in graph embeddings under such conditions. Next in chapter 3 we lay technical definitions of essential arcs, punctured tori and triple torus. These will help us look at lemmas that will be used in proving our main theorem that every 4-representative embedding on the tripe torus contains two NSCs. Non-contractible separating cycles and critical embeddings are looked at in chapter 4. Finally, in chapter 5 we give conclusion and a brief discussion of the findings.

It is found that Ellingman & Zhao's method of proving that every 4-representative embedding on the double tori contains an NSC works efficiently in proving existence of NSCs in 4-representative graph embeddings on the triple tori.

CHAPTER 2

LITERATURE REVIEW

In this chapter we review some literature on conditions for existence of NSCs in graph embeddings and those that have proven existence of NSCs in graph embeddings under such conditions.

Some authors have written on graph embeddings focusing on existence of NSCs of embedded graphs on surfaces. They have explored conditions for existence of NSCs of embedded graphs and some have proven existence of NSCs in graph embeddings under such conditions. Most of these conditions involve representativity or face width of graph embeddings. The representativity of an embedding Ψ is denoted $\rho(\Psi)$ and Ψ is k-representative if $\rho(\Psi) \geq k$.

Ellingman N. & Zha X in [**5**] conjectured that if a graph G is embedded on a surface of genus (orientable or non orientable) at least 2, then it may have a Non contractible separating cycle (NSC). According to this conjecture, a triple torus falls within such surfaces. Since this condition is not sufficient and cannot guarantee existence of NSCs, they argued that sufficient conditions for existence of an NSC in embeddings are of interest.

They defined a suitable surface as one of genus (orientable or non orientable) at least 2. This implies that a triple torus is a suitable surface. Barnette (1980) conjectured that every triangulation in a suitable orientable surface has an NSC. Ellingman N. & Zha X in [**5**], further proposed that any triangulation of the double torus whose shortest Non-contractible cycle with length at least 4 has a Non-contractible separating cycle. This was a corollary of their proven theorem that every 4-representative embedding on the double torus contains a Non-contractible separating cycle.

Zha X in [**15**] conjectured more generally that every 3-representative embedding in a suitable surface, (orientable or non-orientable) has an NSC. Ellingman in [**5**] argued that representativity condition would be best possible here as some authors have given examples of embeddings with representativity 2 and no NSC.

Robertson N. & Thomas R in [**14**], proved that every 3-representative embedding in the Klein bottle has an NSC. Vitray R. P in [**3**], further proved that every 11-representative embedding in a suitable surface has an NSC. Zha in [**13**] reduced the representativity condition to 6-representative for orientable surfaces and 5-representative for non orientable ones. Their orientable result originally required a 7-representative embedding, but it was improved using a suggestion of Vitray R. P, and the result for 6-representative orientable embeddings also appear in Brunet, see [**3**] and [**1**].

Still on existence of NSCs in graph embeddings, Mohar. B in [**9**], proposed that if G is a 3-connected graph embedded with face width at least 3, then all Σ-facial walks are induced NSCs. Brunet R. proved that a graph embedding of representativity w in a suitable orientable surface contains $\lfloor \frac{w-9}{8} \rfloor$ disjoint and pairwise homotopic NSCs. Mohar B. & Thomassen C in [**11**], conjectured that given a triangulation of a surface of genus $g \geqslant 2$ and a number h, $1 \leqslant h \leqslant g-1$ there must be an NSC Γ such that the two surfaces separated by Γ have genus h and $g-h$ respectively. Mohar B. conjectured that the same result holds for any 3-representative embedding, see [**1**] and [**11**].

Vitray R. P in [**3**], conjectured that an embedded graph in an orientable surface with face-width at least 3 will contain a non-trivial surface-separating cycle. From this conjecture it can be easily seen that a cycle C in an embedded graph with facewidth at least 3 is separating if C may be spanned by a collection of facial cycles.

Ellingman N. & Zha X in [**5**], tackled the simplest suitable surface, the double torus. They proved that every 4-representative graph embedding in the double torus has an NSC which separates the surface into 2 connected components. This improved on the best previous condition ($\rho \geq 6$) but does not achieve the goal of Zha's conjecture ($\rho \geq 3$).

In this work, we consider exploring existence of non-contractible separating cycles in critical embeddings on the triple torus. We wish to prove existence of NSCs in 4-representative graphs embedded on the triple torus and we consider the triple torus to be the one in figure 2.1.

Figure 2.1: The triple torus.
Source: Radicliffe.

CHAPTER 3

ESSENTIAL ARCS, PUNCTURED TORI AND TRIPLE TORUS

This chapter presents some technical definitions on essential arcs, punctured tori and triple torus. We also state and prove some lemmas which will be used in the proof of our main theorem.

3.1 Essential arcs and punctured tori

The following lemma shows how two disjoint Non-Contractible circles on a torus splits the surface into 2 cylinders, we refer the reader to [**5**].

Lemma 3.1.1

(i) Two disjoint non-contractible circles in the torus are homotopic (up to orientation) and together they separate the torus into two cylinders.

(ii) Two non-contractible circles on the torus are disjoint (under homotopy) if and only if they are homotopic (up to orientation).

Figure 3.1 summarises the description above where one non-contractible separating cycle on a torus cuts the torus into a cylinder.

When two disjoint non-contractible separating cycles are used, the torus will be separated into two cylinders.

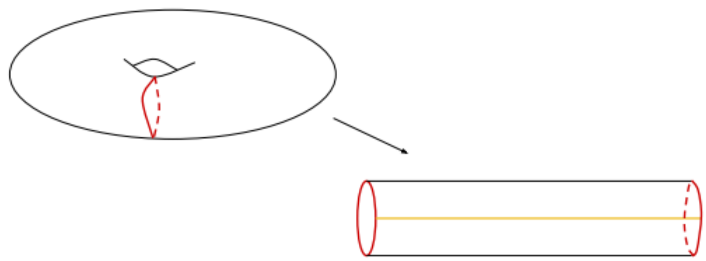


Figure 3.1: Formation of a cylinder from a torus.
Source: Radicliffe.

Suppose Σ_0 is a surface with one boundary circle Γ. Let Σ be a surface without boundary obtained by pasting a disk D along Γ. Suppose P is an arc in Σ_0 joining two distinct points of Γ in Σ_0 with $P^0 \cap \Gamma = \emptyset$. The endpoints of P divide Γ into two subarcs Γ_1 and Γ_2. In that case, we have two circles $P \cup \Gamma_1$, $P \cup \Gamma_2$ which are homotopic. If these two circles are non-contractible then P will be called an essential arc, we refer to [**5**]. Now let $\Sigma_0 = T_0$ and $\Sigma = T$ be tori. Then T_0 is a punctured torus. A punctured torus with one boundary circle Γ is denoted T_0^1 and a punctured torus with two boundary circles Γ_a, Γ_b is denoted T_0^2.

3.2 The Cylinder-Strip partitions

Let P and P' be parallel disjoint essential arcs on a punctured torus. Together they separate the punctured torus into a cylinder and a strip. A cylinder in that case has two boundary circles and a strip has one. This partition is denoted CS(P, P'), See [**5**].

Let Σ_0^2 be a surface with two boundary circles Γ_a and Γ_b and Σ^2 be a surface without boundary obtained by pasting disks D_a and D_b along Γ_a and Γ_b respectively. Suppose P_a is an arc that joins two distinct points of Γ_a in Σ_0^2 with $P_a^o \cap \Gamma_a = \emptyset$. Similarly, suppose P_b is an arc that joins two distinct points of Γ_b in Σ_0^2 with $P_b^o \cap \Gamma_b = \emptyset$. The endpoints of P_a divide Γ_a into two subarcs Γ_a^1 and Γ_a^2, and the two circles $P_a \cup \Gamma_a^1$, $P_a \cup \Gamma_a^2$ are homotopic in Σ^2. Similarly, the endpoints of P_b divide Γ_b into subarcs Γ_b^1, Γ_b^2 and the two circles $P_b \cup \Gamma_b^1$, $P_b \cup \Gamma_b^2$ are homotopic in Σ^2. If both these circles are Non-contractible then both P_a and P_b will be called essential arcs.

Now assume that $\Sigma_0^2 = T_0^2$ is a punctured torus with two boundary circles and $\Sigma^2 = T$ a torus which is not punctured. Suppose that P_a and P_a' are disjoint essential arcs (so that their four endpoints are all disjoint). We say P_a and P_a' are parallel if the end points of P_a are not separated on Γ_a by the end points of P_a'. Similarly, we say P_b and P_b' are parallel if the endpoints of P_b are not separated on Γ_b by the endpoints of P_b'.

In the above case we may label the four endpoints of the pairs of the arcs in order along Γ_a and Γ_b respectively as x_a, y_a, x_a', y_a' with P_a from x_a to y_a, P' from x_a' to y_a'. Similarly the four endpoints in order along Γ_b can be labeled as x_b, y_b, x_b', y_b' with P_b from x_b to y_b, P_b' from x_b' to y_b'. By Lemma 3.1.1 (i), the disjoint homotopic circles $P_a \cup x_a\Gamma_a y_a$ and $P_a' \cup x_a'\Gamma_a y_a'$ separates T into disjoint cylinders C_a, C_a' and the homotopic circles $P \cup x_b\Gamma_b y_b$ and $P_b' \cup x_b'\Gamma_b y_b'$ separate T into disjoint cylinders C_b, C_b'. Let C_a' be the cylinder containing D_a^o. Then C_a' is further separated by $y_a\Gamma_a x_a' \cup y_a'\Gamma_a x_a$ into D_a^o and a disk S_a bounded by $P_a \cup y_a\Gamma_a x_a' \cup P_a' \cup y_a'\Gamma_a x_a$. S_a is called a strip with ends $y_a\Gamma_a x'_a$ and $y'_a\Gamma_a x_a$. Thus, $\Gamma_a \cup P_a \cup P'_a$ separates the torus T into C_a, S_b and D_a, and $P \cup P'$ separates the torus $T_0 = T\backslash D^o_a$ into C_a and S_a. C_a, S_a is called the cylinder-strip partition of T_0 induced by P_a and P'_a and is denoted $CS(P_a, P'_a)$.

Now let C'_b be the cylinder containing D^o_b. Then C'_b is further separated by $y_b\Gamma_b x'_b \cup y'_b\Gamma_b x_b$ into D^o_b and a disk S_b bounded by $P_b \cup y_b\Gamma_b x'_b \cup P'_b \cup y'_b\Gamma_b x_b$. S_b is called a strip with ends $y_b\Gamma_b x'_b$ and $y'_b\Gamma_b x_b$. Thus, $\Gamma_b \cup P_b \cup P'_b$ separates the torus T into C_b, S_b and D_b, and $P_b \cup P'_b$ separates the torus $T_0 = T\backslash D^o_b$ into C_b and S_a. C_b, S_b is called the cylinder-strip partition of T_0 induced by P_b and P'_b and is denoted by $CS(P_b, P'_b)$. Figure 3.2 gives an illustration of the description above.

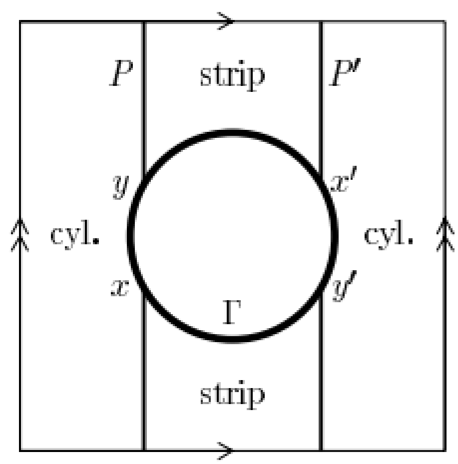


Figure 3.2: Construction of cylinder and strip.

The following lemma shows how 3 essential arcs, two of them being parallel, can be properly placed on the cylinder-strip partition of a punctured torus.

Lemma 3.2.1: The Cylinder-Strip Lemma. Given a punctured torus T^2_0, with two boundary circles Γ_a and Γ_b, let P_a, P'_a and P_b, P'_b be a pair of parallel disjoint essential arcs. Let P''_a be an essential arc disjoint from P_a and P'_a and P''_b be an essential arc disjoint from P_b and P'_b. Then,

(i) Both ends of P''_a must lie in the cylinder, C_a or both ends must lie in the strip,S_a of $C_aS_a(P_a, P'_a)$.

(ii) Both ends of P''_b must lie in the cylinder, C_b or both ends must lie in the strip,S_b of $C_bS_b(P_b, P'_b)$

(iii) If both ends of P''_a lie in the strip S_a, then they lie at opposite ends.

(iv) If both ends of P''_b lie in the strip S_b, then they lie at opposite ends.

Figure 3.3 shows cases where the cylinder-strip lemma is violated, we refer to [**5**].

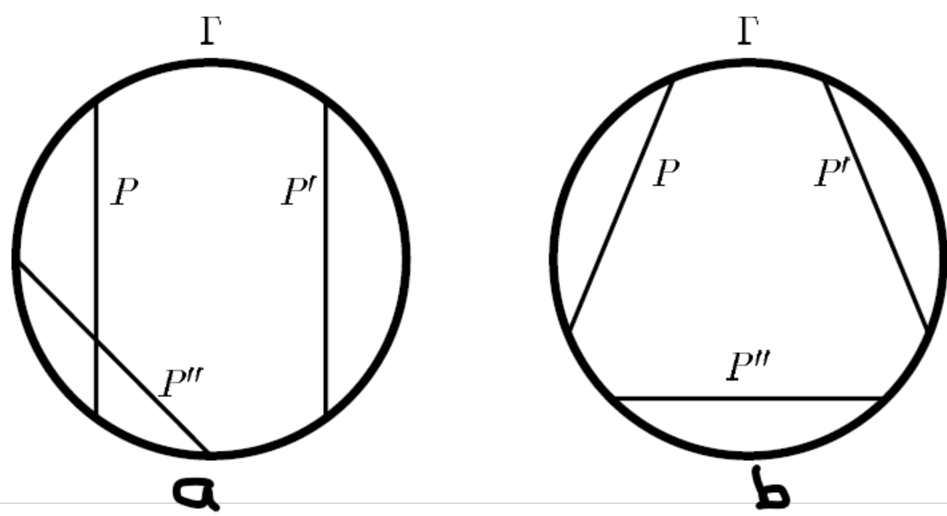


Figure 3.3: (a) violates (i) of cylinder strip lemma, (b) violates (iii) of cylinder strip lemma.

In dealing with essential arcs in a given punctured torus T_0, with boundary Γ, Ellingman N. & Zha X represented Γ as a circle and essential arcs as chords of the circle, see [**5**]. Suppose Γ is a single boundary circle of compact surface Σ_0. Let D be a closed disk in Σ_0 and suppose $\Gamma \cap D = \Gamma \cap \partial D$ consists of a finite number of components. We say that Γ and D intersect essentially if every arc in D joining two distinct components of $\Gamma \cap D$ is essential.

The following lemma shows how many possible components of $\Gamma \cap D$ can occur in a punctured torus.

Lemma 3.2.2: Suppose Γ_a and Γ_b are boundary circles of a punctured torus T_0^2, and Γ_a and Γ_b intersect disks D_a and D_b essentially, respectively. Let $L_a = \partial D_a$ and $L_b = \partial D_b$ both oriented clockwise. Let $\Gamma_a^1, \Gamma_a^2, ..., \Gamma_a^k$ be the components of $\Gamma_a \cap D_a = \Gamma_a \cap L_a$ and $\Gamma_b^1, \Gamma_b^2, ..., \Gamma_b^k$ be the components of $\Gamma_b \cap D_b = \Gamma_b \cap L_b$ both oriented clockwise along their respective boundaries where $\Gamma_a^i = x^i L_a y^i$ and $\Gamma_b^j = x^j L_b y^j$ for each i and j respectively.

(i) $k \leq 4$

(ii) If $k = 2$ then $(y_a^1, x_a^1, y_a^2, x_a^2)$ occur in that clockwise order along Γ_a and $(y_b^1, x_b^1, y_b^2, x_b^2)$ occur in that clockwise order along Γ_b

(iii) If $k = 3$ then $(y_a^1, x_a^1, y_a^2, x_a^2, y_a^3, x_a^3)$ occur in that clockwise order along Γ_a and $(y_b^1, x_b^1, y_b^2, x_b^2, y_b^3, x_b^3)$ occur in that clockwise order along Γ_b

(iv) If $k = 4$ then $(y_a^1, x_a^1, y_a^2, x_a^2, y_a^3, x_a^3, y_a^4, x_a^4)$ occur in that clockwise order along Γ_a and

$(y_b^1, x_b^1, y_b^2, x_b^2, y_b^3, x_b^3, y_b^4, x_b^4)$ occur in that clockwise order along Γ_b

Proof. By expanding D_a and or D_b slightly if necessary, we may assume that $x_a^i \neq y_a^i$

for all i and similarly that $x_b^j \neq y_b^j$ for all j. If we add a disk D_a along Γ_a, L_a and Γ_a are both contractible and have natural clockwise orientations, which must oppose each other where they meet. Thus y_a^i is followed on Γ_a by x_a^i and x_a^i must be followed by some y_a^i. On the other hand, if we add a disk D_b along Γ_b, L_b and Γ_b are both contractible and have natural clockwise orientations, which must oppose each other where they meet. Thus y_b^j is followed on Γ_b by x_b^j and x_b^j must be followed by some y_b^j.

We first prove (ii), (iii) and (iv) and then prove (i)

(ii) When $k = 2$ the given orders are the only possible ones.

(iii) Suppose $k = 3$. There are only two possible clockwise orders along Γ_a. If the order is $(y_a^1, x_a^1, y_a^3, x_a^3, y_a^2, x_a^2)$ then the essential arc $y_a^3 L_a x_a^1$ has both ends at the same end of the strip of $C_a S_a(y_a^1 L_a x_a^2, y_a^2 L_a x_a^2)$ contradicting (iii) of the cylinder-strip lemma and if the order is $(y_b^1, x_b^1, y_b^3, x_b^3, y_b^2, x_b^2)$ then the essential arc $y_b^3 L_b x_b^1$ has both ends at the same end of the strip of $C_b S_b(y_b^1 L_b x_b^2, y_b^2 L_b x_b^2)$ contradicting (iv) of the cylinder-strip lemma.

(iii) By shifting D_a slightly we may apply (iii) separately to both collections Γ_a^1, Γ_a^2, Γ_a^3 and Γ_a^1, Γ_a^3, Γ_a^3 and get the required order. Also by By shifting D_b slightly we may apply (iii) separately to both collections Γ_b^1, Γ_b^2, Γ_b^3 and Γ_b^1, Γ_b^3, Γ_b^3 and get the required order.

(i) If $k \geq 5$, then by shifting the disk D_a slightly we may assume that $k = 5$. By similar reasoning to (iv) the clockwise order must be $(y_a^1, x_a^1, y_a^2, x_a^2, y_a^3, x_a^3, y_a^4, x_a^4, y_a^5, x_a^5)$. Now the essential arc $y_a^4 L_a x_a^5$ has one end in the cylinder and the other end in the strip of $C_aS_a(y_a^5L_ax_a^1, y_a^2L_ax_a^3)$ contradicting (i) of the cylinder-strip lemma. On the other hand, by shifting the disk D_b slightly we may assume that $k = 5$. By similar reasoning to (iv) the clockwise order must be $(y_b^1, x_b^1, y_b^2, x_b^2, y_b^3, x_b^3, y_b^4, x_b^4, y_b^5, x_b^5)$. Now the essential arc $y_b^4L_bx_b^5$ has one end in the cylinder and the other end in the strip of $C_bS_b(y_b^5L_bx_b^1, y_b^2L_bx_b^3)$ contradicting (ii) of the cylinder-strip lemma.

3.3 The double and triple tori

Ellingman N. & Zha X, see [**5**], considered a double torus S_2. Suppose we have an oriented non contractible separating circle Γ of the torus S_2. It separates S_2 into two punctured tori A_0, B_0 which are both closed and include Γ. When convinient, A_0 is completed with disc $A^\star$ to a torus A and B_0 with a disk $B^\star$ to a torus B. If both A and B inherit the orientation of S_2, Γ will be clockwise in A and anticlockwise in B. In other words, Γ goes clockwise around $A^\star$ so $A^\star$ is to the right of Γ in A, and A_0 is to the left of Γ in both A and S_2. Similarly B_0 is to the right of Γ, see figure 3.4.

Now we consider a triple torus S_3. Suppose we have 2 oriented noncontractible separating circles Γ_a, Γ_b of the triple torus. Together they separate the triple torus into three punctured tori, A_0, B_0, C_0. A_0 contains Γ_a, B_0 contains both Γ_a and Γ_b and C_0 contains Γ_b meaning that both A_0, B_0 and C_0 are closed. When convinient, we will complete A_0 with disk $A^\star$ to torus A, B_0 with disks $B_1^\star$ and $B_2^\star$ to torus B and C_0 with disk $C^\star$ to torus C.

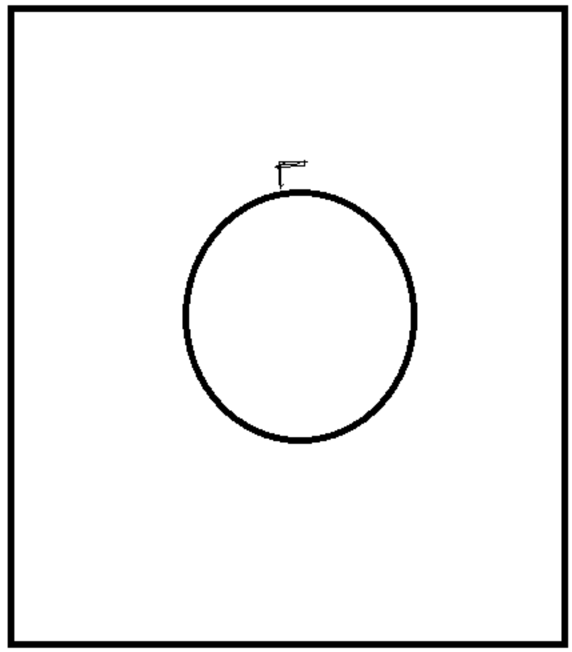


Figure 3.4: NSC Γ on a double torus

A_0 and C_0 are punctured tori with one boundary circle each, Γ_a and Γ_b respectively. B_0 is a punctured torus with two boundary circles Γ_a and Γ_b. We assume that Γ_a goes clockwise around $A^\star$. So $A^\star$ is to the right of Γ_a in A and A_0 is to the left of Γ_a in both A and S_3. Γ_b goes clockwise around $C^\star$ in C so $C^\star$ is to the right of Γ_b in C and C_0 to the left of Γ_b in both C and S_3. Since Γ_a goes clockwise around $A^\star$ in A, it goes anticlockwise in B, meaning B_0 is to the left of Γ_a in that case. Similarly B_0 is to the right of Γ_b, see figure 3.5

We now discuss some of the ways Γ_a and Γ_b can pass through given closed disks D_a and D_b. Let $L_a = \partial D_a$ and $L_b = \partial D_b$. Suppose $\Gamma_a \cap D_a$ has finitely many components

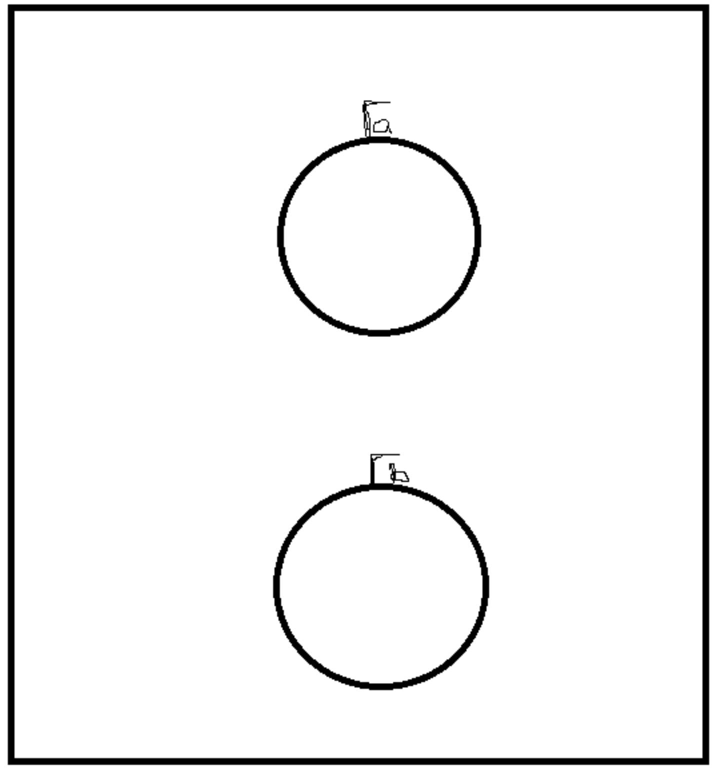


Figure 3.5: NSCs Γ_a and Γ_b on a triple torus

including but not limited to Γ_a^1, Γ_a^2, Γ_a^3, Γ_a^4 and $\Gamma_b \cap D_b$ has finitely many components Γ_b^1, Γ_b^2, Γ_b^3, Γ_b^4 with;

i. Each $\Gamma_a^i \cap L_a$ has at most 2 components, $1 \leq i \leq 4$ and $\Gamma_b^j \cap L_b$ has at most 2 components $1 \leq j \leq 4$

ii. There is an arc P_1 in D_a with ends on L_a and $P_1^o \subset D_a^o$ such that $P_1 \cap \Gamma_a = x_1, x_2, x_3, x_4$ where x_1, x_2, x_3, x_4 are in that order along P_1. Each x_i belongs to Γ_a^i and Γ_a^2 and Γ_a^3 cross (not just intersect) P_1 at x_2 and x_3 respectively.

Similarly there is an arc P_2 in D_b with ends on L_b and $P_2^o \subset D_b$ such that $P_2 \cap \Gamma_b = y_1, y_2, y_3, y_4$ where y_1, y_2, y_3, y_4 are in that order along P_2. Each y_i

belongs to Γ_b^j and Γ_b^2 and Γ_b^3 cross P_2 at y_2 and y_3 respectively.

Assume that $x_1 P_1 x_2 \subset A_0$ and $y_1 P_2 y_2 \subset B_0$. For each i, let ia denote the first component of $\Gamma_a^i \cap L_a$ following Γ_a^i along Γ_a and ib the last. And for each j let jd denote the first component of $\Gamma_b^j \cap L_b$ following Γ_b^j along Γ_b and jf the last.

By the fact that Γ_a is separating and using orientations of A_0 and B_0 the components $1a$, $2b$, $3a$, $4b$, $4a$, $3b$, $2a$, $1b$ occur in that clockwise order along L_a (so $2a \neq 2b, 3a \neq 3b$, but possibly $1a = 1b$ or $4a = 4b$). Similarly by the fact that Γ_b is separating and using the orientations of B_0 and C_0 the components $1d$, $2e$, $3d$, $4e$, $4d$, $3e$, $2d$, $1e$ occur in that clockwise order along L_b (so $2e \neq 2d, 3d \neq 3e$ but possibly $1d = 1e$, or $4e = 4d$)

There are six possible cyclic orders in which components Γ_a^1, Γ_a^2, Γ_a^3, Γ_a^4 can occur along Γ_a and six possible cyclic orders in which components Γ_b^1, Γ_b^2, Γ_b^3, Γ_b^4 can occur along Γ_b. In each case they occur in pairs which are equivalent up to reversal of Γ_a and Γ_b respectively. If we know that Γ_a intersects the (closures of the) components of $D_a \backslash \Gamma_a$ essentially, then for some of these orders we can place restrictions on where additional components of $\Gamma_a \cap D_a$ can be. Similarly if we know that Γ_b intersects the (closures of the) components of $D_b \backslash \Gamma_b$ essentially, then for some of these orders we can place restrictions on where additional components of $\Gamma_b \cap D_b$ can be.

CHAPTER 4

NON-CONTRACTIBLE SEPARATING CYCLES AND CRITICAL EMBEDDINGS

This chapter focuses on how new NSCs can be constructed from old ones. We also discuss a method called augmentation, which raises representativity of embeddings from 4 to 6 in this chapter and then we define critical embeddings. That will take us to the proof of our main theorem in the same chapter.

The following lemma shows possible orders of arrangements of segments of $\Gamma_a \cap D_a$ and $\Gamma_b \cap D_b$ on a triple torus.

Lemma 4.1: Suppose Γ_a and Γ_b are two non-contractible separating circles of S_3 with disks D_a and D_b and components Γ_a^1, Γ_a^2, Γ_a^3, Γ_a^4 of $\Gamma_a \cap D_a$ and components Γ_b^1, Γ_b^2, Γ_b^3, Γ_b^4 of $\Gamma_b \cap D_b$ as described above. Suppose further that Γ_a intersects the closure of every component of $D_a \backslash \Gamma_a$ essentially (in A_0 or B_0 as appropriate) and that Γ_b intersects the closure of every component of $D_b \backslash \Gamma_b$ essentially (in B_0 or C_0 as appropriate).

1. If the components occur along Γ_a in the order (1432) then $\Gamma_a \cap (2aL_a 1b)^o = \Gamma_a \cap (4aL_a 3b)^o = \emptyset$ and if the components occur along Γ_b in the order (1432) then $\Gamma_b \cap (2dL_b 1e)^o = \Gamma_b \cap (4dL_b 3e)^o = \emptyset$

2. If the components occur along Γ_a in the order (1342) then $\Gamma_a \cap (2aL_a 1b)^o = \Gamma_a \cap (3aL_a 4b)^o = \emptyset$ and if the components occur along Γ_b in the order (1342) then $\Gamma_b \cap (2dL_b 1e)^o = \Gamma_b \cap (3dL_b 4e)^o = \emptyset$

Proof: Suppose in Case 1 that $\Gamma_a \cap (2aL_a 1b)^o \neq \emptyset$. Let Γ_a^5 (or for short 5) be the component of $\Gamma \cap (2aL_a 1b)^o$ closest to 1b on L_a. By lemma 3.2.2 (iii), $5 \subset (2b\Gamma_a 1a)^o$. But then $5L_a 1b$ violates $C_a S_a(1aL_a 2b, 3aL_a 4b)(i)$ (i,e violates Lemma 3.2.1 (i)). Here it is necessary if $\Gamma_a^1 = 1a = 1b$ is a single point. Thus, $\Gamma_a \cap (2aL_a 1b)^o = \emptyset$. Similarly suppose that $\Gamma_b \cap (2dL_b 1e)^o \neq \emptyset$. Let Γ_b^5 be the component of $\Gamma_b \cap (2dL_b 1e)^o$ closest to 1e on L_b. By lemma 3.2.2 (iii), $\Gamma_b^5 \subset (2e\Gamma_b 1d)^o$. But then $5L1e$ violates $C_b S_b(1dL_b 2e, 3dL_b 4e)(ii)$(i.e violates Lemma 3.2.1 (ii)). Here it is necessary if $\Gamma_b^1 = 1d = 1e$ is a single point. Thus $\Gamma_b \cap (2dL_b 1e)^o = \emptyset$.

Similarly suppose in Case 2 that $\Gamma_a \cap (2aL_a 1b) \neq \emptyset$. Let Γ_a^5 be the component of $\Gamma_a \cap (2aL_a 1b)^o$ closest to 1b on L_a. By lemma 3.2.2 (iii), $5 \subset (2b\Gamma_a 1a)^o$. But then $5L_a 1b$ violates $C_a S_a(1aL_a 2b, 4aL_a 3b)(i)$(i.e violates Lemma 3.2.1 (i)). (Note: We assume that essential arcs can be shifted slightly if necessary to apply the Cylinder-Strip Lemma). Here it is necessary if $\Gamma_a^1 = 1a = 1b$ is a single point. Thus, $\Gamma_a \cap (2aL_a 1b)^o = \emptyset$. On the other hand suppose that $\Gamma_b \cap (2dL_b 1e)^o \neq \emptyset$. Let Γ_b^5 be the component of $\Gamma_b \cap (2dL_b 1e)^o$ closest to 1e on L_b. By lemma 3.2.2 (iii), $\Gamma_b^5 \subset$

$(2e\Gamma_b 1d)^o$. But then $5L1e$ violates $C_bS_b(1dL_b2e, 4dL_b3e)(ii)$(i.e violates Lemma 3.2.1 (ii)). Here it is necessary if $\Gamma_b^1 = 1d = 1e$ is a single point. Thus $\Gamma_b \cap (2dL_b1e)^o = \emptyset$. The rest of the proof is similar.

4.1 Construction of Non-Contractible Separating Circles

Next we discuss a method of constructing new noncontractible separating circles from old ones on general surfaces in theorem 4.1.1 and Corollary 4.1.2 will show how this applies to a triple torus.

Theorem 4.1.1: Let Σ be a surface with two oriented noncontractible separating circles Γ_a and Γ_b separating the surface into 3 closed components A_0, B_0 and C_0. Suppose there are sections Γ_a^1, Γ_a^2, Γ_a^3, Γ_a^4 of Γ_a in that order along Γ_a, and sections Γ_b^1, Γ_b^2, Γ_b^3, Γ_b^4 of Γ_b in that order along Γ_b. Suppose further that there are arcs P_{12}, P_{34} in A_0, Q_{23}, Q_{41} in B_0, also, Q_{12}, Q_{34} in B_0 and R_{23}, R_{41} in C_0, such that;

(i) Γ_a^1, Γ_a^2, Γ_a^3, Γ_a^4 are disjoint, so are Γ_b^1, Γ_b^2, Γ_b^3, Γ_b^4

(ii) P_{12}^o, P_{34}^o, Q_{23}^o, Q_{41}^o are disjoint from Γ_a and Q_{12}^o, Q_{34}^o, R_{23}^o, R_{41}^o are disjoint from Γ_b

(iii) P_{12} has ends a_1, a_2; P_{34} has ends a_3, a_4; Q_{23} has ends b_2, b_3; Q_{41} has ends b_4, b_1 where a_i, b_i are two ends of each Γ_a^i (not necessarily in order along Γ_a); Q_{12} has ends c_1, c_2; Q_{34} has ends c_3, c_4; R_{23} has ends d_2, d_3 and R_{41} has ends d_4, d_1; where c_j, b_j are two ends of each Γ_b^j (not necessarily in order along Γ_b).

(iv) $P_{12} \cup P_{34}$ separates A_0 into a component A_1 with boundary $P_{12} \cup a_2\Gamma_a a_3 \cup P_{34} \cup a_4\Gamma_a a_1$ (one circle) and a component A_2 with boundary $(a_1\Gamma_a a_2 \cup P_{12}) \cup (a_3\Gamma_a a_4 \cup P_{34})$ (two circles). $Q_{23} \cup Q_{41}$ similarly separates B_0 into a component B_1 with boundary $Q_{23} \cup b_3\Gamma_a b_4 \cup Q_{41} \cup b_1\Gamma_a b_2$ (one circle) and a component B_2 with boundary $(b_2\Gamma_a b_3 \cup Q_{23}) \cup (b_4\Gamma_a b_1 \cup Q_{41})$ (two circles). In the same way, $Q_{12} \cup Q_{34}$ separates B_0 into a component B_3 with boundary $Q_{12} \cup c_2\Gamma_b c_3 \cup Q_{34} \cup c_4\Gamma_b c_1$ (one circle) and a component B_4 with boundary $(c_1\Gamma_b c_2 \cup Q_{12} \cup (c_3\Gamma_b c_4 \cup Q_{34})$ (two circles), while $R_{23} \cup R_{41}$ similarly separates C_0 into a component C_1 with boundary $R_{23} \cup d_3\Gamma_b d_4 \cup R_{41} \cup d_1\Gamma_b d_2$ (one circle) and a component C_2 with boundary $(d_2\Gamma_b d_3 \cup R_{23}) \cup (d_4\Gamma_b d_1 \cup R_{41})$ (two circles). Then;

$\Gamma' = \Gamma_a^1 \cup P_{12} \cup \Gamma_a^2 \cup Q_{23} \cup \Gamma_a^3 \cup P_{34} \cup \Gamma_a^4 \cup Q_{41}$ and

$\Gamma'' = \Gamma_b^1 \cup Q_{12} \cup \Gamma_b^2 \cup R_{23} \cup \Gamma_b^3 \cup Q_{34} \cup \Gamma_b^4 \cup R_{41}$ are also Non-contractible separating circles in Σ^2, separating $A_1 \cup B_2$ from $A_2 \cup B_1$ and $B_3 \cup C_2$ from $B_4 \cup C_1$ respectively.

Proof. We observe from the conditions of the theorem that Γ' separates $A_1 \cup B_2$ from $A_2 \cup B_1$ and that Γ'' separates $B_3 \cup C_2$ from $B_4 \cup C_1$. It is enough to show that both Γ' and Γ'' are Non-contractible or equivalently that none of $A_1 \cup B_2$, $A_2 \cup B_1$,$B_3 \cup C_2$ and $B_4 \cup C_1$ is homeomorphic to a disk.

(i) Since A_1 has one boundary circle, it is homeomorphic to a disk with handles and or crosscaps attached. Since B_2 has two boundary circles, it is homeomorphic to a cylinder with handles and or crosscaps attached. If A_1 is just a disk and B_2 just a cylinder, the way they are attached along the segments of Γ_a between Γ_a^2 and Γ_a^3 and between Γ_a^4 and Γ_a^1 means that the result would be homeomorphic to a punctured torus. More generally, the result is homeomorhic to a punctured torus with handles and or crosscaps added, which is not a disk. Therefore $A_1 \cup B_2$ is not a disk.

(ii) Since B_1 has one boundary circle, it is homeomorphic to a disk with handles and or crosscaps attached. Since A_2 has two boundary circles, it is homeomorphic to a cylinder with handles and or crosscaps attached. If B_1 is just a disk and A_2 just a cylinder, the way they are attached along the segments of Γ_a between Γ_a^2 and Γ_a^3 and between Γ_a^4 and Γ_a^1 means that the result would be homeomorphic to a punctured torus. More generally, the result is homeomorphic to a punctured torus with handles and or crosscaps added, which is not a disk. Therefore $A_2 \cup B_1$ is not a disk.

(iii) Since B_3 has one boundary circle, it is homeomorphic to a disk with handles and or crosscaps attached. Since C_2 has two boundary circles, it is homeomorphic to a cylinder with handles and or crosscaps attached. If B_3 is just a disk and C_2 just a cylinder, the way they are attached along the segments of Γ_b between Γ_b^2 and Γ_b^3 and between Γ_b^4 and Γ_b^1 means that the result would be homeomorphic to a punctured torus. More generally, the result is homeomorphic to a punctured torus with handles and or crosscaps added, which is not a disk. Therefore $B_3 \cup C_2$ is not a disk.

(iv) Since C_1 has one boundary circle, it is homeomorphic to a disk with handles and or crosscaps attached. Since B_4 has two boundary circles, it is homeomorphic to a cylinder with handles and or crosscaps attached. If C_1 is just a disk and B_4 just a cylinder, the way they are attached along the segments of Γ_b between Γ_b^2 and Γ_b^3 and between Γ_b^4 and Γ_b^1 means that the result would be homeomorphic to a punctured torus. More generally, the result is homeomorhic to a punctured torus with handles and or crosscaps added, which is not a disk. Therefore $B_4 \cup C_1$ is not a disk

Since (i) and (ii) proves that components $A_1 \cup B_2$ and $A_2 \cup B_1$ are not disks, then

$\Gamma' = \Gamma_a^1 \cup P_{12} \cup \Gamma_a^2 \cup Q_{23} \cup \Gamma_a^3 \cup P_{34} \cup \Gamma_a^4 \cup Q_{41}$

is Non-contractible. Similarly, since (iii) and (iv) proves that components $B_3 \cup C_2$ and $B_4 \cup C_1$ are not disks, then

$\Gamma'' = \Gamma_b^1 \cup Q_{12} \cup \Gamma_b^2 \cup R_{23} \cup \Gamma_b^3 \cup Q_{34} \cup \Gamma_b^4 \cup R_{41}$ is also Non-contractible.

We now apply this to the triple torus, with weaker versions of conditions (ii) and (iv), and stating condition (iv) in a way that is specific to the triple torus.

Corollary 4.1.2. Suppose Γ_a and Γ_b are two Non-contractible separating circles in the triple torus S_3, separating S_3 into three (closed) punctured tori A_0, B_0 and C_0. Suppose there are sections Γ_a^1, Γ_a^2, Γ_a^3, Γ_a^4 of Γ_a in that order along Γ_a, and sections Γ_b^1, Γ_b^2, Γ_b^3, Γ_b^4 of Γ_b in that order along Γ_b, arcs P_{12}, P_{34} in A_0; Q_{23}, Q_{41} in B_0 ; Q_{12},

Q_{34} in B_0 and R_{23}, R_{41} in C_0 such that;

(i) Γ_a^1, Γ_a^2, Γ_a^3, Γ_a^4 are disjoint, so are Γ_b^1, Γ_b^2, Γ_b^3, Γ_b^4

(ii) P_{12}^o, P_{34}^o, Q_{23}^o, Q_{41}^o are disjoint from Γ_a and Q_{12}^o, Q_{34}^o, R_{23}^o, R_{41}^o are disjoint from Γ_b

(iii) P_{12} has ends a_1, a_2; P_{34} has ends a_3, a_4; Q_{23} has ends b_2, b_3 and Q_{41} has ends b_4, b_1 where a_i, b_i are two ends of each Γ_a^i (not necessarily in order along Γ_a) and Q_{12} has ends c_1, c_2, Q_{34} has ends c_3, c_4, R_{23} has ends d_2, d_3 and R_{41} has ends d_4, d_1 where c_j, b_j are two ends of each Γ_b^j (not necessarily in order along Γ_b).

(iv) P_{12}, P_{34} are homotopic with endpoints fixed in A_0 to a pair of parallel essential arcs and Q_{23}, Q_{41} are homotopic with endpoints fixed in B_0 to a pair of parallel essential arcs. Similarly Q_{12}, Q_{34} are homotopic with endpoints fixed in B_0 to a pair of parallel essential arcs and R_{23}, R_{41} are homotopic with endpoints fixed in C_0 to a pair of parallel essential arcs. Then;

$\Gamma' = \Gamma_a^1 \cup P_{12} \cup \Gamma_a^2 \cup Q_{23} \cup \Gamma_a^3 \cup P_{34} \cup \Gamma_a^4 \cup Q_{41}$ and

$\Gamma'' = \Gamma_b^1 \cup Q_{12} \cup \Gamma_b^2 \cup R_{23} \cup \Gamma_b^3 \cup Q_{34} \cup \Gamma_b^4 \cup R_{41}$ are also Non-contractible separating circles in S_3.

Proof: Conditions (ii) and (iv) mean that by shifting Γ_a slightly, we can make P_{12}, P_{34} and Q_{23}, Q_{41} into pairs of parallel essential arcs. Similarly, by shifting Γ_b slightly we can make Q_{12}, Q_{34} and R_{23}, R_{41} into pairs of parallel essential arcs. Then each of the (slightly shifted) punctured tori is separated into a cylinder and a strip in each case. Now applying Theorem 4.1.1 completes the proof.

Sets of arcs P_{12}, P_{34}, Q_{23}, Q_{41} and Q_{12}, Q_{34}, R_{23}, R_{41} satisfying Corollary 4.1.2 are called orthogonal arrangements of parallel arcs, or OP for short. They are are referred to as $OP(P_{12}, P_{34}; Q_{23}, Q_{41})$ and $OP(Q_{12}, Q_{34}; R_{23}, R_{41})$. The arcs P_{12}, P_{34}, Q_{23}, Q_{41} are not required to have interiors disjoint from Γ_a, just from sections Γ_a^1, Γ_a^2, Γ_a^3, Γ_a^4. Similarly, the arcs Q_{12}, Q_{34}, R_{23}, R_{41} are not required to have interiors disjoint from Γ_b, but just from sections Γ_b^1, Γ_b^2, Γ_b^3, Γ_b^4. The most common case of this is illustrated in figure 4.1. When forming an OP, the two hatched essential arcs joined by a hatched segment of Γ may be considered equivalent to the single dashed essential arc, as long as the hatched segment of Γ_a does not intersect $\Gamma_a^1, \Gamma_a^2, \Gamma_a^3, \Gamma_a^4$.

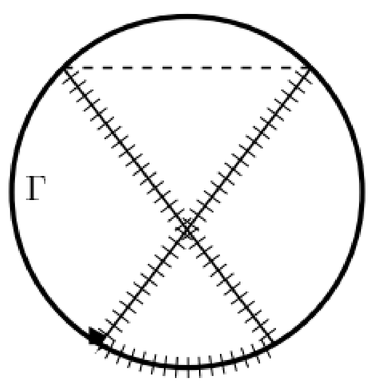


Figure 4.1: Equivalent to an essential arc, see [**5**].

4.2 Critical embeddings

In proving the existence of Non-contractible separating cycles (NSCs), Ellingman used an approach of examining embeddings which are very close to having an

NSC.

The following lemma justifies existence of NSCs in critical embeddings with critical edges.

Lemma 4.2.1 : Let Σ be a suitable surface (one with genus at least 2), and $k \geq 3$ (the representativity of an embedding on such a surface). Suppose there is a k-representative embedding in Σ that does not have an NSC. Then there exists a k-representative embedding Ψ of a simple 2-connected graph in Σ that does not have an NSC, with a face f containing non-adjacent vertices x, y so that when the edge xy is inserted across the face f, $\Psi^+ = \Psi \cup xy$ has an NSC. We call Ψ a critical embedding with critical edge xy, see [**5**].

Proof: Let Ψ_0 be a k-representative embedding of a graph G_0 in Σ with no NSC. Since $k \geq 3$ and since multiple edges bounding a disk can be reduced to a single edge without affecting existence of an NSC, we may assume that G_0 is simple. Moreover, by reducing it to essential 2-component, we may assume that G_0 is 2-connected.

Ellingman N. & Zha X, see[**5**], defined an augmentation of an embedding to be either;

(i) The addition of an edge across a face between two nonadjacent vertices on that face or,

(ii) If every face is a triangle as in triangulation (bounded by a 3-cycle), then in some face (uvw) subdivide one edge uv with a new vertex x and then add the edge ux

Neither (i) nor (ii) decreases the representativity. In a sequence of augmentations, any augmentation following one of type (ii) must be of type (i).

If we apply a sequence of augmentations to Ψ_0, each embedding is k-representative with a graph that is simple and 2-connected. Moreover, by applying a sequence of augmentations to Ψ_0 we can increase its representativity arbitrarily. First, we complete Ψ_0 to a triangulation using type (i) augmentations. It is well known that in a triangulation, representativity equals the length of a shortest noncontractible cycle. Given an edge $e = vw$ on a shortest noncontractible cycle, belonging to two triangles (uvw) and (twv), we can apply four augmentations of type (ii), (i), (ii), (i) with the effect of deleting vw, adding two new vertices x_1, x_2, and adding paths vx_1w, vx_2w, ux_1x_2t. This destroys all shortest noncontractible cycles through e without creating any new shortest noncontractible cycles. After destroying all shortest noncontractible cycles in this way, the representativity must increase by at least one, then we can repeat the process. The essence of repeating the process is to raise representativity to 6 at which point Zha & Zhao proved that an NSC exists, see [**13**].

Therefore, it is possible to apply a sequence of augmentations to Ψ_0 to raise the representativity to at least 6, at which point an NSC exist. Let Ψ be an embedding in the sequence before the first augmentation that creates an NSC. Type (ii) augmentations cannot create an NSC if one does not already exist. So the augmentations is of type (i) and the result follows.

4.3 Main theorem

In this section, it is shown that every embedding on the triple torus with representativity at least 4 contains two NSCs. We begin with a standard result on 4-representative graphs.

Definition 4.3.1: Equivalence of embeddings: Two embeddings $\Psi_a : G \longrightarrow \Sigma$ and $\Psi_b : G \longrightarrow \Sigma$ are equivalent, denoted by $\Psi_a \backsimeq \Psi_b$ if there exists a graph automorphism $\alpha : G \longrightarrow G$ and an orientation preserving surface homeomorphism $\eta : \Sigma \longrightarrow \Sigma$ for which the map compositions $\Psi_a \circ \eta : G \longrightarrow \Sigma$ and $\alpha \circ \Psi_b : G \longrightarrow \Sigma$ are identical, see [**10**].

The following lemma shows how two, 4-representative graph embeddings on the triple torus are equivalent.

Lemma 4.3.2: Let $\Psi_a : G \longrightarrow S_3$ and $\Psi_b : G \longrightarrow S_3$ be two embeddings of a graph G into the triple torus S_3. Suppose there exists an automorphism $\alpha : G \longrightarrow G$ between G and itself. Then the embeddings $\Psi_a : G \longrightarrow S_3$ and $\Psi_b : G \longrightarrow S_3$ are equivalent.

Proof: First we show that there is an orientation preserving surface homeomorphism $\eta : S_3 \longrightarrow S_3$. [Bernadi. O (2011)] argued that a map is a connected graph embedded on a surface, considered up to orientation preserving homeomorphism.

This is obvious since a triple torus is always homeomorphic to itself. For easy identification of the preservation of the orientation, we consider the case where the homeomorphism η is an identity map.

Finally we show that the map compositions $\Psi_a \circ \eta : G \longrightarrow S_3$ and $\alpha \circ \Psi_b : G \longrightarrow S_3$ are identical. Let e be an arc with end points v, w on S_3, then $\eta(e) = e$ since η is identity map. Since Ψ_a is an embedding, $\Psi_a((u, v)) = e$. Similarly, let e be an edge of graph G with end points u, v, i.e, edge (u, v). Then since Ψ_b is an embedding, $\Psi_b((u, v)) = e$ where e is an arc on S_3. Applying η on e we have $\eta(e) = e$ since e is an identity map. Hence, the two embeddings Ψ_a and Ψ_b are equivalent.

The lemma below shows the existence of a disk D which contains the union of f and all faces that share at least one vertex with f, given that f is the face of an embedding which has representativity at least 3. The subsequent lemma shows how this applies to 4-representative embeddings on the triple torus.

Lemma 4.3.3: Let f be a face of an embedding $\Psi : G \to \Sigma$, where Σ is not a 2-sphere. Let k be an integer such that $\rho(\Psi, f) > 2k + 1$ and let $B_k(f)$ be the union of f and all faces that share at least one vertex with f. Then there is a disk $D_k(f) \subset \Sigma$ which contains $B_k(f)$ such that $\partial D_k(f) \subseteq \partial B_k(f)$, see [**14**].

Proof: Any contractible simple closed curve $\gamma \in \Sigma$ bounds a unique disc since Σ is not a 2-sphere. Denote this disc by $int(\gamma)$. Clearly, any disc containing $B_k(f)$ must contain the disk $int(\gamma)$ for any simple closed curve γ in $B_k(f)$. Let

$D_k = B_k(f) \cup \{int(\gamma) \mid \gamma$ is a simple closed curve in $B_k(f)\}$.

Each closed curve γ in D_k is homotopic to some closed curve contained in $B_k(f)$ since any part of γ in $int(\gamma)$ can be moved by homotopy to the boundary of $int(\gamma)$, which is contained in $B_k(f)$. D_k is simply connected and it is also connected by construction and $\partial D_k \subseteq \partial B_k(f)$. Since the only simply connected compact surfaces are the 2-sphere and the closed disk, it suffices to show that D_k is a 2-manifold with boundary. By construction it follows that D_k is closed. Moreover D_k is a union of closed faces. Therefore, a singularity can only appear at the vertex of the embedded graph. But by the following reason a true singularity is excluded. If g and h are faces in $B_k(f)$ meeting at a vertex x, let γ be a closed curve starting at a point in $int(f)$ leading to g going through x to h, and returning to f, such that $\mid z \in S^1 \mid \gamma(z) \in \Psi(G) \mid \leq 2k+1$. Since $\rho(f) > 2k+1$, γ bounds a disk in D_k. Consequently all the faces at x which lie between g and h (one or the other side) also lie in D_k.

Lemma 4.3.4: Let f_a and f_b be faces of two 4-representative embeddings Ψ_a and Ψ_b which are equivalent on the triple torus, and let F_a be the union of f_a and all faces that share at least one vertex with f_a. Similarly let F_b be the union of f_b and all faces that share at least one vertex with f_b.

(i) The face f_a is a disk D_a^1 with boundary cycle L_a^1, and there is a disk $D_a^2 \supset D_a^1$ with boundary cycle L_a^2, such that $F \subseteq D_a^2$ and $L_a^2 = \partial D_a^2 \subseteq \partial F_a$. Similarly, the face f_b is a disk D_b^1 with boundary cycle L_b^1, and there is a disk $D_b^2 \supset D_b^1$ with boundary cycle L_b^2, such that $F \subseteq D_b^2$ and $L_b^2 = \partial D_b^2 \subseteq \partial F_b$

(ii) Any path P_a in D_a^2 with both ends on L_a^2 must be a segment of L_a^2 or must contain a vertex of L_a^1 and any path P_b in D_b^2 with both ends on L_b^2 must be a segment of L_b^2 or must contain a vertex of L_b^1

Proof.

i (i) is a special case of Lemma 4.3.3.

ii If (ii) fails, there would be a path P_a in D_a^2 internally disjoint from L_a^2 joining two vertices of L_a^2 and not intersecting L_a^1. Labeling the ends a, b of P_a appropriately, $P_a \cup bL_a^2 a$ would separate $(aL_a^2 b)^o$ from L_a^1. But this contradicts the fact that since $L_a^2 \subseteq \partial F_a$, every point of L_a^2 has an arc joining it to L_a^1 that does not intersect the graph except at its endpoints. Similarly, there would be a path P_b in D_b^2 internally disjoint from L_b^2 joining two vertices of L_a^2 and not intersecting L_b^1. Labeling the ends d, e of P_b appropriately, $P_b \cup dL_b^2 e$ would separate $(dL_b^2 e)^o$ from L_b^1. But this contradicts the fact that since $L_b^2 \subseteq \partial F_b$, every point of L_b^2 has an arc joining it to L_b^1 that does not intersect the graph except at its endpoints.

We now present the main theorem.

Theorem 4.3.5: Every 4-representative embedding on the triple torus contains two non-contractible separating cycles which splits the triple torus into 3 connected components.

Proof: Suppose the theorem is false. By Lemma 4.2.1, there are critical 4-representative embeddings Ψ_a and Ψ_b (of simple 2-connected graphs) with no NSC, while $\Psi_a^+ =$

$\Psi_a \cup vw$ and $\Psi_b^+ = \Psi_b \cup xy$ have NSCs. Suppose that vw is added across the face f_a and xy across the face f_b. Let D_a^1, D_a^2, L_a^1, L_a^2 and let D_b^1, D_b^2, L_b^1, L_b^2 be as provided by Lemma 1.8 for both f_a and f_b and let $L_a = L_a^1 \cup L_a^2$ and $L_b = L_b^1 \cup L_b^2$.

Every NSC in Ψ_a^+ must contain the edge vw and every NSC in Ψ_b^+ must contain the edge xy. Of all NSCs in Ψ_a^+, let Γ_a be the one that minimises $\| \Gamma_a \cap D_a^2 \|$ (the number of components in $\Gamma_a \cap D_a^2$) and subject to this also minimises $\| \Gamma_a \cap D_a^1 \|$. Similarly let Γ_b be the one that minimises $\| \Gamma_b \cap D_b^2 \|$ (the number of components in $\Gamma_b \cap D_b^2$) and subject to this also minimises $\| \Gamma_b \cap D_b^1 \|$.

Then each component of $\Gamma_a \cap D_a^2$ contains at most one component of $\Gamma_a \cap D_a^1$ (and using lemma 4.1 (ii)), at most 2 components of $\Gamma_a \cap L_a^2$. By the same argument, each component of $\Gamma_b \cap D_b^2$ contains at most one component of $\Gamma_b \cap D_b^1$ (and using lemma 4.1 (ii)), at most 2 components of $\Gamma_b \cap L_b^2$. We often abbreviate Γ_a^i to i. (In later parts of the proof we also use components of Γ_a', abbreviated i', where $i = 1, 2, ...$ and components of Γ_b', abbreviated j', where $j = 1, 2, ...$). We represent subsegments of component i as ij where j is a letter, e.g $3a$ is a subsegment of $3 = \Gamma_a^3$.

Since Ψ_a and Ψ_b are equivalent by lemma 4.3.2, all operations on Γ_a applies similarly on Γ_b. We now focus on Γ_a and conclude to operations of Γ_b by equivalence of the embeddings.

The minimality assumption further guarantees that any arc in D_a^2 that joins different components Γ_a^i, Γ_a^j of $\Gamma \cap D_a^2$ and is otherwise disjoint from Γ_a is essential. If it were not essential then we could replace one of the segments $i\Gamma_a j$ or $j\Gamma_a i$ with a section of L_a^2, reducing $\parallel \Gamma_a \cap D_a^2 \parallel$. This is true even if the segment of L_a^2 we wish to use, intersects other components of $\Gamma_a^2 \cap D_a^2$, because those other components must also be part of the segment of Γ_a we are replacing.

Let Γ_a^3 be the component of $\Gamma_a \cap D_a^2$ that contains vw. Then $\Gamma_a^3 \cap L_a$ has four components which we name $3a$, $3b$, $3c$, $3d$ in order along Γ_a with $3a, 3d \subset L_a^2$ and $3b, 3c \subset L_a^1$. We may assume that $v \in 3b$ and $w \in 3c$. For ease of description, we assume that D_a^2 is drawn as a circular disk and Γ_a^3 passes downwards through D_a^2, with $3a$ containing its top point and $3d$ its bottom point. Other than Γ_a^3, no other component of $\Gamma_a \cap D_a^2$ contains more than one component of $\Gamma_a \cap L_a^1$.

In fact, any other component Γ_a^i of $\Gamma_a \cap D_a^2$ is one of two types. If $\parallel \Gamma_a^i \cap D_a^1 \parallel = 1$, i is a segment of L_a^2. Otherwise, $\parallel \Gamma_a^i \cap L_a \parallel = 3$ and i includes two segments ia, ic of L_a^2 and one segment ib of L_a^1 with ia, ib, ic in that order along Γ_a. Since Ψ_a is critical, Γ_a intersects both $(3bL_a^1 3c)^o$ and $(3cL_a^1 3b)^o$, otherwise we could reroute Γ_a to avoid (the interior of) $3b\Gamma_a 3c = vw$. Moreover, each component of $\Gamma_a \cap D_a^2$ which intersects $(3cL_a^1 3b)^o$ cannot be rerouted via $3dL_a^2 3a$ or we could reduce $\parallel \Gamma_a \cap D_a^1 \parallel$.

Let Γ_a^2 denote any such component, with $2a, 2c \subset L_a^2$ and $2b \subset L_a^1$. Note that Γ_a^2 passes upwards through D_a^2, so that the half of f_a to the right of Γ_a^3 is also to the right of Γ_a^2. Since Γ_a^2 cannot be rerouted there is at least one component of $\Gamma_a \cap D_a^2$ that intersects $(2aL_a^2 2c)^o$. Let Γ_a^1 denote any such component, which must be a segment of L_a^2 and passes downwards through D_a^2. In similar way, we can find Γ_a^4 passing upwards through D_a^2, intersecting $(3aL_a^2 2c)^o$ at $4a$ and $4c$ and intersecting $(3bL_a^1 3c)^o$ at $4b$. Then we must also have $\Gamma_a^5 = 5$ passing downwards through D_a^2 and contained in $(4cL_a^2 4a)^o$. In general, it is not known whether a given component of $\Gamma_a \cap L_a$ is trivial (single vertex) or not. The above description can be well represented by figure 4.2.

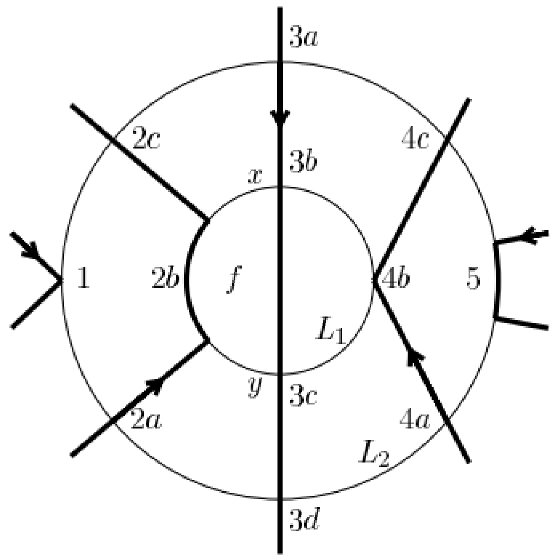


Figure 4.2: Segments of Γ in D_2, see [**5**].

Let A_0 denote the part of S_3 to the left of Γ_a, and B_0 the part to the right; Also B_0 is the part of S_3 to the left of Γ_b and C_0 to the right. A_0, B_0 and C_0 are punctured tori. For $i \geq 1$, let A_i denote the unique component of $A_0 \cap D_a^2$ to which Γ_a^i belongs; define B_i similarly. (If $\Gamma_a^i \subset L_a^2$, one of A_i or B_i will be just Γ_a^i itself). We know that $A_1 = A_2$, $B_2 = B_3$, $A_3 = A_4$ and $B_4 = B_5$.

We will frequently use the orthogonal arrangements of parallel paths to construct a new NSC Γ_a' in Ψ_a^+. In notation, $OP(P, P'; Q, Q')$, P, P' will be paths in A_0 and Q,

Q' those in B_0. There are two common ways in which this provides a contradiction. First, Γ'_a may avoid the edge vw, and so be an NSC for Ψ_a; we indicate this by $AOP(P, P'; Q, Q')$. Second, $\Gamma'_a \cap D^2_a$ may have fewer components than $\Gamma_a \cap D^2_a$; we indicate this by $COP(P, P'; Q, Q')$.

Suppose P, P', Q, Q' all lie in D^2_a. When we form Γ'_a from Γ_a we delete the interiors of four nontrivial segments of Γ_a, say S_1, S_2, S_3, S_4 and then add the interiors of P, P', Q, Q'. Each end of S_j lies in some component of $\Gamma_a \cap D^2_a$. Suppose each S_j intersects s_j components of $\Gamma_a \cap D^2_a$, then $s_j \geq 1$. When we delete S^o_j, the number of components in D^2_a changes by $2 - s_j$. When we add P^o, P'^o, Q, Q'^o, the number of components in D^2_a changes by -4. Thus $\| \Gamma'_a \cap D^2_a \| = \| \Gamma_a \cap D^2_a \|$ $+4 - s_1 - s_2 - s_3 - s_4$.

The above analysis is valid even when the interiors of P, P', Q, Q' intersect components of $\Gamma_a \cap D^2_a$. If we do not have $s_1 = s_2 = s_3 = s_4 = 1$, then we have $COP(P, P'; Q, Q')$. In particular, let $OOP[i](P, P'; Q, Q')$ denote the situation in which some component i of $\Gamma_a \cap D^2_a$ contains an odd number of the eight endpoints of $P, P'; Q, Q'$ (counted with multiplicity). Then $s_j > 1$ for some j, so this is a special case of $COP(P, P'; Q, Q')$.

Now we break into cases according to the order of 1,2,3,4,5 along Γ_a. For any distinct components i_1, i_2,...,i_k of $\Gamma_a \cap D^2_a$, we say that Γ_a has $(i_1 i_2, ..., i_k)$ if i_1, i_2,...,i_k occur in that order along Γ_a.

(A) Suppose Γ_a has (1432). (Since we do not mention component 5, no assumption is made about its position). By lemma 4.1, $\Gamma_a \cap (2aL_a^2 1)^o = \Gamma_a \cap (4aL_a^2 3d)^o = \emptyset$. Note that by lemma 3.2.2, $\Gamma_a \cap (2cL_a^2 3a)^o \subset (3\Gamma_a 2)^o$.

For easy understanding of the subsequent arguments in the proof, figure 4.3 shows how we construct new NSCs using corollary 4.1.2 in the first two cases here. The solid chords are essential paths in A_0, the dashed chords are essential paths in B_0, and the edges of Γ used by the new NSC are hatched.

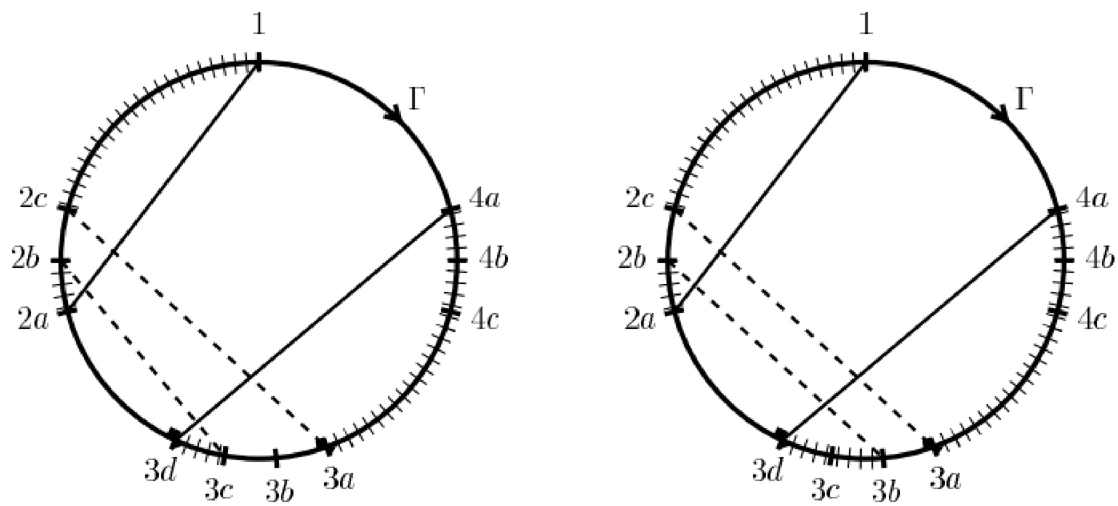


Figure 4.3: Construction of new NSCs in the first 2 cases of A. Case 1 (left), case 2 (right), see [**5**].

First, suppose that $\Gamma_a \cap (3cL_a^1 2b)^o = \emptyset$. Let $P = 2c(\partial B_3)3a$ (P is just $2cL_a^2 3a$ if $\Gamma_a \cap (2cL_a^2 3a)^o = \emptyset$). Since $\Gamma_a \cap (2cL_a^2 3a)^o$ is contained in $(3\Gamma_a 2)^o$, we have $P^o \cap \Gamma_a \subset (3\Gamma_a 2)^o$. Now we have $AOP(2aL_a^2 1, 4aL_a^2 3d; 3cL_a^1 2b, P)$. This is illustrated on the left of figure 11. Note that $(3\Gamma 2)^o$ is not hatched, showing that this part of Γ may be used by P if necessary.

Second, suppose that $\Gamma_a \cap (2bL_a^1 3b)^o = \emptyset$. Since $\Gamma_a^1 \cap (2cL_a^2 3a)^o \subset (3\Gamma_a 2)^o$, we may have $OOP[1](2aL_a^2 1, 4aL_a^2 3d; 2cL_a^2 3a, 2bL_a^1 3b)$. This is illustrated on the right of figure 4.3. Though these two cases appear similar, they are different.

Finally, we may suppose that there exists $\Gamma_a^6 = 6$ that intersects $(2bL_a^1 3b)^o$ and $\Gamma_a^7 = 7$ that intersects $(3cL_a^1 2b)^o$. By lemma 3.2.2, 2,3,6 and 7 are the only components of $\Gamma_a \cap D_a^2$ intersecting B_3, and Γ_a has (3627). If Γ_a has (271) then we have $OOP[1](2aL_a^2 1, 4aL_a^2 3d; 2bL_a^1 6b, 3cL_a^1 7b)$. If Γ_a has (173) then we have $OOP[1](2aL_a^2 1, 4aL_a^2 3d; 2bL_a^2 6b, 7bL_a^1 2b)$. By symmetry we may also exclude the cases where Γ_a has (1234), (5234), or (5432).

(B) Suppose Γ_a has (1342). By Lemma 4.1, $\Gamma_a \cap (2aL_a^2 1)^o = \Gamma_a \cap (3aL_a^2 4c)^o = \emptyset$. Suppose that $\Gamma_a \cap (2cL_a^2 3a)^o = \emptyset$, then we have $OOP[1](2aL_a^2 1, 3aL_a^2 4c; 2cL_a^2 3a, 2bL_a^1 3b)$. Therefore, we may assume $\Gamma_a \cap (2cL_a^2 3a) \neq \emptyset$, and similarly $\Gamma_a \cap (3dL_a^2 2a)^o \neq \emptyset$. Let $\Gamma_a^6 = 6$ intersect $(2cL_a^2 3a)^o$, and $\Gamma_a^7 = 7$ intersect $(3dL_a^2 2a)^o$. By lemma 3.2.2, 2,3,6,7 are the only components of $\Gamma_a \cap D_a^2$ intersecting B_3 and Γ_a has (3627). Then we have $OOP[1](2aL_a^2 1, 3aL_a^2 4c; 2cL_a^2 6, 3dL_a^2 7)$. Figure 4.4 shows how new NSCs are constructed in this case.

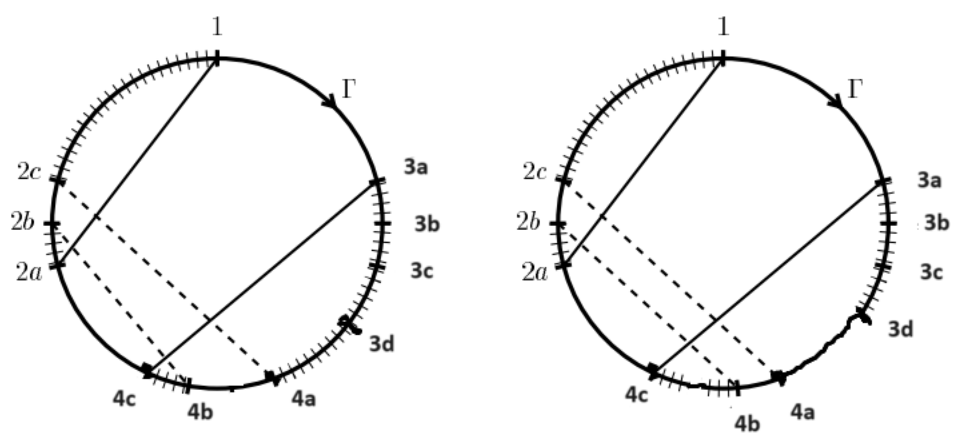


Figure 4.4: Construction of new NSCs in the first 2 cases of B. Case 1 (left), case 2 (right)

By symmetry, we may also exclude the cases where Γ_a has (1243), (5324) or (5423).

Now we know that Γ_a must have either (1324) or (1423), and either (5243) or (5342). So the overall order must be (13524) or (14253). These cases are symmetric, so let us assume the order is (14253). Given this order, there is a symmetry that reverses Γ_a and swaps A_0 and B_0. For our standard picture of D_a^2, this amounts to rotating D_a^2 by 180^o and reversing Γ_a.

Consider the components of $\Gamma_a \cap D_a^2$ that intersect $(2aL_a^2 2c)^o$. Each such component lies in $3\Gamma_a 4)^o$, otherwise we could choose that component as 1 and have case (A) or (B). Let 1 be the first and $1'$ the last such component along $3\Gamma_a 4$. (Possibly $1 = 1'$). By Lemma 3.2.2 there are at most three such components, and 1 is the first and $1'$ the last, along $2aL_a^2 2c$. Thus, $\Gamma_a \cap (2aL_a^2 1)^o = \Gamma(1'L_a^2 2c)^o = \emptyset$. If there are three distinct components 1, $1^\star$, $1'$ in order along Γ_a, then $3aL_a^2 4c$ violates $CS(1L_a^2 1^\star, 1'L_a^2 2c)(i)$ in A_0. Therefore, there are at most two such components. Similarly, at most two components of $\Gamma_a \cap D_a^2$ intersects $(4cL_a^2 4a)^o$, they lie in $(2\Gamma_a 3)^o$ and if 5 is the first and $5'$ the last along $2\Gamma_a 3$ (possibly 5=5′), then $\Gamma_a \cap (4cL_a^2 5')^o = \Gamma_a \cap (5L_a^2 4a)^o = \emptyset$.

If $\Gamma_a \cap (2cL_a^2 3a)^o \neq \emptyset$, we denote the component of $\Gamma_a \cap D_a^2$ closest to 3a by 6, and that closest to 2c by $6'$ (possibly 6=6′). If $\Gamma_a \cap (4aL_a^2 3d)^o \neq \emptyset$, we denote the component of $\Gamma_a \cap D_a^2$ closest to 2a by 7 and that closest to 3d by $7'$ (possibly 7=7′). By Lemma 3.2.2, Γ_a has $(366'277')$ suitably modified to identify components that are the same and delete components that do not exist. Similarly, if $\Gamma_a \cap (3aL_a^2 4c)^o \neq \emptyset$, we denote the component of $\Gamma_a \cap D_a^2$ closest to 3a by 8 and that closest to 4c by $8'$ (possibly 8=$8'$). If $\Gamma_a \cap (4aL_a^2 3d)^o \neq \emptyset$ we denote the component of $\Gamma_a \cap D_a^2$ closest to 4a by 9 and that closest to 3d by $9'$ (possibly 9=$9'$). By Lemma 3.2.2, Γ_a has $(388'499')$ suitably modified. Note that $6, 6', 7, 7', 8, 8', 9, 9'$ may or may not intersect L_a^1.

Claim 1. At least one of 7 and 8 exists

Proof. If not, we have the $OP(3bL_a^1 4b, 3aL_a^2 4c; 3cL_a^1 2b, 3dL_a^2 2a)$ which produces Γ_a' with $\| \Gamma_a' \cap D_a^2 \| = \| \Gamma_a \cap D_a^2 \|$ and $\| \Gamma_a' \cap D_a^1 \| = \| \Gamma_a \cap D_a^1 \| -2$ contradicting the minimallity of Γ_a.

Claim 2. At most one of 6 and 7 exists. By symmetry, at most one of 8 and 9 exists.

Proof. Suppose both 6 and 7 exists. By Lemma 3.2.2(i), 2, 3, 7 are the only components of $\Gamma_a \cap D_a^2$ intersecting B_3, and Γ_a has (3627). To avoid an arc (not necessarily path) from 4 to 5 in B_5 violating $CS(6L_a^2 3a, 7L_a^2 2a)(i)$ in B_0, Γ_a must have (275) when it has (374), and must have (573) when it has (462). So Γ_a has either (364275) or (346257).

Case (2.1) Suppose Γ_a has (364275). If 8 does not exist, let $P = 3aL_a^2 4c$ and $P' = 3bL_a^1 4b$; If 9 does not exist, let $P = 4aL_a^2 3d$ and $P' = 4bL_a^1 3c$. In either case we have $OOP[2](P, P'; 3dL_a^2 7, 2cL_a^2 6)$. Therefore, 8 and 9 exist. By lemma 1.3, 3,4,8,9 are the only components of $\Gamma_a \cap D_a^2$ intersecting A_3, and Γ_a has (3849).

To avoid an arc from 1 to 2 in A_1 violating $CS(3aL_a^2 8, 4aL_a^2 9)(i)$ in A_0, Γ_a must have (429) when it has (318), and must have (923) when it has (814). So Γ_a has either (318429) or (381492). If Γ_a has (318429) we have $OOP[2](8L_a^2 4c, 9L_a^2 3d; 6L_a^2 3a, 7L_a^2 2a)$. If Γ_a has (381492) we have $OOP[5](8L_a^2 4c, 9L_a^2 3d; 6L_a^2 3a, 5L_a^2 4a)$.

Case (2.2) Suppose Γ_a has (346257). If $\Gamma_a \cap (3aL_a^2 4c)^o = \emptyset$ let $P = 3aL_a^2 4c$ and $P' = 3bL_a^1 4b$; if $\Gamma_a \cap (4aL_a^2 3d)^o = \emptyset$ let $P = 4aL_a^2 3d$ and $P' = 4bL_a^2 3c$. In either case we have $OOP[5](P, P'; 3dL_a^2 7, 5L_a^2 4a)$.

Claim 3. If 7 exists, then $7 = 7'$ and Γ_a has (275). By symmetry, if 8 exists then $8 = 8'$ and Γ_a has (1′84).

Proof. We first show that Γ_a does not have (364). Suppose Γ_a has (364). Since at most one of 8 and 9 exists by Claim 2, we may take paths P, P' to be either $3aL_a^2 4c$, $3bL_a^1 4b$ or $4bL_a^1 3c$, $4aL_a^1 3d$. Then we have $OOP[5](P, P'; 3dL_a^2 7', 5L_a^2 4a)$.

If $7 \neq 7'$ then to avoid an arc from 4 to 5 violating $CS(3dL_a^2 7', 7L_a^2 2a)(i)$ in B_0, Γ_a must have $(2757'3)$ and hence $(57'3)$. Thus $7 = 7'$, and since Γ_a does not have $(57'3) = (573)$, it must have (275).

Claim 4. If 6 exists then $6 = 6'$ and Γ has (462). By symmetry, if 9 exists then $9 = 9'$ and Γ_a has (492).

Proof. We first show that Γ_a does not have (364). Suppose Γ_a has (364). Since at most one of 8 and 9 exists by Claim 2, we may take paths P, P' to be either $3aL_a^2 4c$, $3bL_a^2 4b$ or $4bL_a^1 3c$, $4aL_a^2 3d$. Then we have $OOP[5](P, P'; 6L_a^2 3a, 5L_a^2 4a)$.

If $6 \neq 6'$ then to avoid an arc from 4 to 5 violating $CS(2cL_a^2 6', 6L_a^2 3a)(i)$ in B_0, Γ_a must have $(3646'2)$ and hence (462).

Now from claim 1 we may assume without loss of generality that 7 exists. By claim 2, 6 does not exist, and at most one of 8 or 9 exists. If 8 exists then Γ_a has $(31'84275)$ by claim 3, and we get $OOP[1'](1'L_a^2 2c, 8L_a^2 4c; 5L_a^2 4a, 7L_a^2 2a)$. If 9 exists, then Γ_a has $(31'49275)$ by claim 4 and we get $OOP[1'](1'L_a^2 2c, 4aL_a^2 9; 5L_a^2 4a, 7L_a^2 2a)$. Therefore, none of 6, 8 or 9 exists.

To summarise; Γ_a has (314275), 7 exists, $7 = 7'$, and none of 6, 8, or 9 exists. To find new NSCs in this situation, we use paths that may lie outside disk D_a^2. By Lemma 4.3.4 (ii), every edge of L_a^2 belongs to a face , contained in D_a^2 that includes a vertex of L_a^1. Applying this to an edge of L_a^2 with at least one end in 1, we obtain a face g with at least one vertex v_1 of 1 and at least one vertex v_2 of 2b. The only components of $\Gamma_a \cap D_a^2$ that g may intersect are 1, 2 and (if $1 = 1'$) $1'$. By lemma 4.3.4 (ii), $g \cap 1$ and $g \cap 1'$ have at most one component each. Apply lemma 4.3.4 to g, letting E_1 and E_2 be the disks, with boundaries M_1 and M_2.

Let O_{12} be an arc from v_1 to v_2 inside g, and let O_{34} be an arc from an interior point v_3 of vw to a vertex v_4 of 4b inside $f_a \cap A_0$. Cut A_0 along O_{12} and O_{34}; the result is a disk with clockwise boundary (in compressed notation).

$$v_1 O_{12} v_2 \Gamma_a^{-1} v_4 O_{34}^{-1} v_3 \Gamma_a^{-1} v_2 O_{12}^{-1} v_1 \Gamma_a^{-1} v_3 O_{34} v_4 \Gamma_a^{-1} v_1$$

(We do not distinguish between two copies of $O_{12}, O_{34}, v_1, v_2, v_3, v_4$ since it will be clear which one we mean). This disk contains a slightly smaller disk R, whose boundary we divide into left, right, top and bottom segments for convenience. Reading bottom to top, R has $Q_1 = v_1\Gamma_a^{-1}3d(L_a^2){-}14a\Gamma_a^{-1}v_1$ on the left and $Q_2 = v_2\Gamma_a 3aL_a^2 4c\Gamma_a v_2$ on the right. Reading left to right, R has O_{12} on the top and O_{12}^{-1} on the bottom. We use $>$, $<$, $\geq$, $\leq$ to denote order along Q_1 or Q_2, so that, for example, $u > v$ means u is above v. Along Q_1 we have $v_1 < 3d < 4a < v_1$, with $4a < 1' < v_1$ if $1' \neq 1$. Along Q_2 we have $v_2 < 2c < 7 < 5 < 3a < 4c < 2a < v_2$.

Now examining $(M_1 \cup M_2) \cap R$, there must be vertices $x_1 > x_2 > x_3 > x_4$ on Q_1, $y_1 > y_2 > y_3 > y_4$ on Q_2 and paths P_1, P_2, P_3, P_4 in R such that

(i) P_1, P_2, P_3, P_4 are vertex-disjoint, except that P_1 and P_2 may intersect at either both of v_1, v_2;

(ii) Each P_i has ends x_i, y_i and is otherwise disjoint from ∂R;

(iii) P_1 and P_4 are segments of M_1, while P_2 and P_3 are segments of M_2; and

(iv) $x_1 \in 1$, $y_1 \in 2$, $x_4 \in 1$ or $1'$, $y_4 \in 2$

The paths P_1 and P_4 are just the obvious segments of $\partial g = M_1$. If M_2 did not contain two disjoint paths P_2, P_3 as described, then we could find a circle in E_2 that was Non-contractible in S_3, contradicting the fact that E_2 is a disk.

If an end of P_2 or P_3 belongs to $(4aL_a^2 3d)^o$ or $(3aL_a^2 4c)^o$, then the path is not essential because it does not have both ends on Γ_a. However it can be extended to essential path in more than one way. Given $x_i > 3d$, define X_i^+ to be $4aL_a^2 x_i$ if $x_i < 4a$ or x_i if $x_i \geqslant 4a$. Given $x_i < 4a$, define X_i^- to be $x_i L_a^2 3d$ if $x_i > 3d$ or x_i otherwise. Define

Y_i^+ and Y_i^- on the right similarly based on the relationship of y_i to $4c$ and $3a$.

Let w_5 be the last vertex of 5 along Γ_a. Note that $5L_a^2 4a = w_5 L_a^2 4a$. Suppose first that $x_3 < 4a$. Then necessarily $x_4 \in 1$. If $y_3 < w_5$ then $OP(P_1, X_3^- \cup P_3; 5L_a^2 4a, 7L_a^2 2a)$ produces a separating cycle which does not use 4b; replace vw by $vL_a^1 w$ to obtain a separating cycle avoiding vw. If $w_5 \leqslant y_3 < 4c$, then since $x_4 \in 1$ we have $AOP(X_3^- \cup P_3 \cup Y_3^-, P_4; 2cL_a^2 3a, 3dL_a^2 7 \cup 7 \cup 7L_a^2 2a)$. If $y_3 \geqslant 4c$ then we have the rather complicated $AOP(P_1, X_3^- \cup P_3 \cup y_3 \Gamma_a^{-1} 4c \cup 4c(L_a^2)^{-1} 3a; 2cL_a^2 3a, 3dL_a^2 7 \cup 7\Gamma_a 5 \cup 5L_a^2 4a)$.

Now suppose that $x_3 \geqslant 4a$, and that $y_2 \leqslant 3a$. If $\Gamma_a \cap (3cL_a^1 2b)^o = \emptyset$ then we have $AOP(P_2, P_3; 2cL_a^2 3a, 3cL_a^2 2b)$. Otherwise 7 must intersect $3cL_a^1 2b$ so 7 has segments 7a, 7c, on L_a^2 and 7b on L_a^1. Then we have $AOP(3aL_a^2 4c, 4bL_a^1 3c; 3cL_a^1 7b, 3dL_a^2 7a)$.

Finally, suppose that $X_3 \geqslant 4a$ and $y_2 > 3a$. If $x_4 \in 1$, then $OP(P_2 \cup Y_2^+, P_4; 5L_a^2 4a, 7L_a^2 2a)$ produces a separating cycle that does not use 4b; replace vw by $vL_a^1 w$ to obtain a separating cycle avoiding vw. If $x_4 \notin 1$ then $1 \neq 1'$ and $x_4 \in 1'$. Let $P_2' = P_2 \cup y_2 \Gamma_a^{-1} 4c \cup 4c(L_a^2)^- 3a$ if $y_2 \geqslant 4c$ and $P_2' = P_2 \cup Y_2^-$ if $3a < y_2 < 4c$. Then we have $AOP(P_4, P_2^-; 2cL_a^2 3a, 3dL_a^2 7 \cup 7\Gamma_a 5 \cup 5L_a^2 4a)$.

Since the embeddings Ψ_a and Ψ_b are equivalent by lemma 4.3.2, all operations applying on Γ_a also applies similarly on Γ_b. We have covered all cases, so this concludes the proof.

CHAPTER 5

CONCLUSION AND RECOMMENDATIONS

This chapter presents conclusion of the study and a brief recommendation. We also suggest possible areas of further study from our results.

5.1 Conclusion

In the conclusion, we comfortably state that the general objective of proving that every 4-representative graph embedding on the triple torus contains two NSCs which separates the triple torus into 3 connected components is achieved. This was done by analysing Ellingman & Zhao's method of proving that every 4-representative embedding on the double torus contains an NSC which splits the torus into two connected components.

After analysing Ellingman & Zhao's method and applying it to the proof of our main theorem we it is found that the method works efficiently and our results are consistent with those obtained by them in their proof.

Therefore it is concluded here that every 4-representative graph embedding on the triple torus contains two NSCs.

The notions of equivalence in graph embeddings and that of homeomorphism of surfaces as defined topologically were used in extending Ellingman & Zhao's method to apply on a triple torus.

5.2 Recommendations

We find that Ellingman & Zhao's method of proving that every 4-representative embedding on the double tori contains an NSC works efficiently in proving existence of NSCs in graph embeddings on the triple tori. As it has been done in this study, by employing equivalence of embeddings and homeomorphism of surfaces it can therefore be recommended that the method works efficiently in proving existence of NSCs in triple toroidal embeddings and possibly in embeddings of connected sums of tori. We recommend that our results can be used to prove existence of NSCs in 4-representative embeddings on connected sum of tori, and even to other higher surfaces.

5.3 Future work/Areas of further research

This study of existence of NSCs in critical triple toroidal embeddings sets a foundation for some future areas of study. Our results can motivate further study on existence of NSCs in graph embeddings on connected sum of n tori for $n \geq 4$.

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Representativity and Non-Contractible Separating Cycles of embeddings on the Triple Torus.

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Abstract

We consider Ellingman and Zhao's method of proving that every 4-representative graph embedding on the double torus contains a Non-Contractible Separating Cycle (NSC). In literature it has not been proven yet that 4-representative embeddings on triple tori contains (NSCs). We use existing theorems including Ellingman and Zhao's method to prove existence of NSCs in embeddings on the triple torus. We find that every 4- representative embedding on the triple torus contains two NSCs which separates the triple torus into 3 connected components, namely punctured tori, two of them with one boundary circle and one with two boundary circles.

Keywords: Non-contractible Separating Cycle (NSC), embeddings, representativity, torus, homeomorphism.

1 Introduction

A graph G is a pair of sets, $V(G)$ and $E(G)$, where $V(G)$ is nonempty and $E(G)$ is a set of 2-element subsets of $V(G)$. A walk in the graph $G = (V, E)$ is a finite sequence of the form $v_{i_0}, e_{j_1}, v_{i_1}, e_{j_2}, \ldots, e_{j_k}, v_{i_k}$, which consists of alternating vertices and edges of G. The walk starts at a vertex. A walk is open if $v_{i_0} \neq v_{i_k}$, see [**5**, **1**]. [1] Graphs can be studied on the sphere (plane), or on other surfaces. A surface is a compact two-dimensional manifold, possibly with boundary. Equivalently, a surface is a compact topological space that is Hausdorff (any two distinct points have disjoint neighbourhoods) and such that every point has a neighbourhood homeomorphic to a plane or a closed half plane, we refer to [**5**].

An embedding Ψ of a graph G on a surface Σ is a crossing free drawing of G on Σ. It maps the vertices of G to distinct points of Σ and its edges to paths of Σ whose endpoints are images of their incident vertices. Representativity of an embedding is defined as a set $\rho(\Psi) = \min\{|\Gamma \cap G| : \Gamma$ is a Non-contractible simple closed curve on $\Sigma\}$. The embedding is critical if it is close to having NSC, see [**6**, **2**].

T is used to denote the torus. A graph G is toroidal if G embeds in T . Let Ψ be an embedding of $G = G(\Psi)$ in T. The closure of each connected component of $T \backslash G(\Psi)$ is called a face of Ψ (closed faces are mostly preferred to open ones). The

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face set of an embedding Ψ in T is denoted by $F(\Psi)$. If the graph is 2-connected and $\rho(\Psi) \geqslant 2$ then each face f is bounded by a cycle, called a facial cycle and is denoted by ∂f. ∂X denotes the boundary of a set $X \subset T$. Two vertices x and y are cofacial by a face f if $x, y \in \partial f$. Embeddings with $\rho(\Psi) \geqslant 4$ have all faces bounded by cycles in graphs, see [**6**].

Some authors have explored conditions for existence of NSCs of embedded graphs and some have proven existence of NSCs in graph embeddings under such conditions. Most of these conditions involve representativity or face width of graph embeddings. Ellingman N. & Zha X in [**2**] conjectured that if a graph G is embedded on a surface of genus (orientable or non orientable) at least 2, then it may have a Non contractible separating cycle (NSC). They proved that every 4-representative graph embedding in the double torus has an NSC which separates the surface into 2 connected components.

In this work we explore existence of non-contractible separating cycles in critical embeddings on the triple torus. We wish to prove existence of NSCs in 4-representative graphs embedded on the triple torus.

2 Results

Proposition 2.1: The Cylinder-Strip Lemma. Given a punctured torus T_0, with two boundary circles Γ_a and Γ_b, let P_a, P'_a and P_b, P'_b be a pair of parallel disjoint essential arcs. Let P''_a be an essential arc disjoint from P_a and P'_a and P''_b be an essential arc disjoint from P_b and P'_b. Then,

(i) Both ends of P''_a must lie in the cylinder, C_a or both ends must lie in the strip,S_a of $C_aS_a(P_a, P'_a)$.

(ii) Both ends of P''_b must lie in the cylinder, C_b or both ends must lie in the strip,S_b of $C_bS_b(P_b, P'_b)$

(iii) If both ends of P''_a lie in the strip S_a, then they lie at opposite ends.

(iv) If both ends of P''_b lie in the strip S_b, then they lie at opposite ends.

Let D be a closed disk in Σ_0 (a punctured surface) and suppose $\Gamma \cap D = \Gamma \cap \partial D$ consists of a finite number of components. We say that Γ and D intersect essentially if every arc in D joining two distinct components of $\Gamma \cap D$ is essential. The following lemma shows how many possible components of $\Gamma \cap D$ can occur in a punctured torus.

Lemma 2.2: Suppose Γ_a and Γ_b are boundary circles of a punctured torus T_0, and Γ_a and Γ_b intersect disks D_a and D_b essentially, respectively. Let $L_a = \partial D_a$ and

$L_b = \partial D_b$ both oriented clockwise. Let $\Gamma_a^1, \Gamma_a^2, ..., \Gamma_a^k$ be the components of $\Gamma_a \cap D_a = \Gamma_a \cap L_a$ and $\Gamma_b^1, \Gamma_b^2, ..., \Gamma_b^k$ be the components of $\Gamma_b \cap D_b = \Gamma_b \cap L_b$ both oriented clockwise along their respective boundaries where $\Gamma_a^i = x^i L_a y^i$ and $\Gamma_b^j = x^j L_b y^j$ for each i and j respectively.

(i) $k \leq 4$

(ii) If $k = 2$ then $(y_a^1, x_a^1, y_a^2, x_a^2)$ occur in that clockwise order along Γ_a and $(y_b^1, x_b^1, y_b^2, x_b^2)$ occur in that clockwise order along Γ_b

(iii) If $k = 3$ then $(y_a^1, x_a^1, y_a^2, x_a^2, y_a^3, x_a^3)$ occur in that clockwise order along Γ_a and $(y_b^1, x_b^1, y_b^2, x_b^2, y_b^3, x_b^3)$ occur in that clockwise order along Γ_b

(iv) If $k = 4$ then $(y_a^1, x_a^1, y_a^2, x_a^2, y_a^3, x_a^3, y_a^4, x_a^4)$ occur in that clockwise order along Γ_a and

$(y_b^1, x_b^1, y_b^2, x_b^2, y_b^3, x_b^3, y_b^4, x_b^4)$ occur in that clockwise order along Γ_b

Proof. By expanding D_a and or D_b slightly if necessary, we may assume that $x_a^i \neq y_a^i$ for all i and similarly that $x_b^j \neq y_b^j$ for all j. If we add a disk D_a along Γ_a, L_a and Γ_a are both contractible and have natural clockwise orientations, which must oppose each other where they meet. Thus y_a^i is followed on Γ_a by x_a^i and x_a^i must be followed by some y_a^i. On the other hand, if we add a disk D_b along Γ_b, L_b and Γ_b are both contractible and have natural clockwise orientations, which must oppose each other where they meet. Thus y_b^j is followed on Γ_b by x_b^j and x_b^j must be followed by some y_b^j.

We first prove (ii), (iii) and (iv) and then prove (i).

(ii) When $k = 2$ the given orders are the only possible ones.

(iii) Suppose $k = 3$. There are only two possible clockwise orders along Γ_a. If the order is $(y_a^1, x_a^1, y_a^3, x_a^3, y_a^2, x_a^2)$ then the essential arc $y_a^3 L_a x_a^1$ has both ends at the same end of the strip of $C_a S_a(y_a^1 L_a x_a^2, y_a^2 L_a x_a^2)$ contradicting (iii) of the cylinder-strip lemma and if the order is $(y_b^1, x_b^1, y_b^3, x_b^3, y_b^2, x_b^2)$ then the essential arc $y_b^3 L_b x_b^1$ has both ends at the same end of the strip of $C_b S_b(y_b^1 L_b x_b^2, y_b^2 L_b x_b^2)$ contradicting (iv) of the cylinder-strip lemma.

(iii) By shifting D_a slightly we may apply (iii) separately to both collections Γ_a^1, Γ_a^2, Γ_a^3 and Γ_a^1, Γ_a^3, Γ_a^3 and get the required order. Also by By shifting D_b slightly we may apply (iii) separately to both collections Γ_b^1, Γ_b^2, Γ_b^3 and Γ_b^1, Γ_b^3, Γ_b^3 and get the required order.

(i) If $k \geq 5$, then by shifting the disk D_a slightly we may assume that $k = 5$. By similar reasoning to (iv) the clockwise order must be $(y_a^1, x_a^1, y_a^2, x_a^2, y_a^3, x_a^3, y_a^4, x_a^4, y_a^5, x_a^5)$. Now the essential arc $y_a^4 L_a x_a^5$ has one end in the cylinder and the other end in the strip of $C_aS_a(y_a^5L_ax_a^1, y_a^2L_ax_a^3)$ contradicting (i) of the cylinder-strip lemma. On the other hand, by shifting the disk D_b slightly we may assume that $k = 5$. By similar reasoning to (iv) the clockwise order must be $(y_b^1, x_b^1, y_b^2, x_b^2, y_b^3, x_b^3, y_b^4, x_b^4, y_b^5, x_b^5)$. Now the essential arc $y_b^4L_bx_b^5$ has one end in the cylinder and the other end in the strip of $C_bS_b(y_b^5L_bx_b^1, y_b^2L_bx_b^3)$ contradicting (ii) of the cylinder-strip lemma.

2.1 The double and triple tori

Suppose we have an oriented non contractible separating circle Γ of the torus S_2. It separates S_2 into two punctured tori A_0, B_0 which are both closed and include Γ. When convinient, A_0 is completed with disc $A^\star$ to a torus A and B_0 with a disk $B^\star$ to a torus B. If both A and B inherit the orientation of S_2, Γ will be clockwise in A and anticlockwise in B. In other words, Γ goes clockwise around $A^\star$ so $A^\star$ is to the right of Γ in A, and A_0 is to the left of Γ in both A and S_2. Similarly B_0 is to the right of Γ. See figure 1.

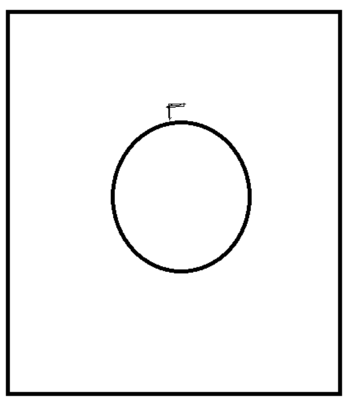

Figure 1: NSC Γ on a double torus

Now we consider a triple torus S_3. Suppose we have 2 oriented noncontractible separating circles Γ_a, Γ_b of the triple torus. Together they separate the triple torus into three punctured tori, A_0, B_0, C_0. A_0 contains Γ_a, B_0 contains both Γ_a and Γ_b and C_0 contains Γ_b meaning that both A_0, B_0 and C_0 are closed. When convinient, we will complete A_0 with disk $A^\star$ to torus A, B_0 with disks $B_1^\star$ and $B_2^\star$ to torus B and C_0 with disk $C^\star$ to torus C.

A_0 and C_0 are punctured tori with one boundary circle each, Γ_a and Γ_b respectively. B_0 is a punctured torus with two boundary circles Γ_a and Γ_b. We assume that Γ_a goes clockwise around $A^\star$. So $A^\star$ is to the right of Γ_a in A and A_0 is to the left of Γ_a

in both A and S_3. Γ_b goes clockwise around $C^\star$ in C so $C^\star$ is to the right of Γ_b in C and C_0 to the left of Γ_b in both C and S_3. Since Γ_a goes clockwise around $A^\star$ in A, it goes anticlockwise in B, meaning B_0 is to the left of Γ_a in that case. Similarly B_0 is to the right of Γ_b. See figure 2.

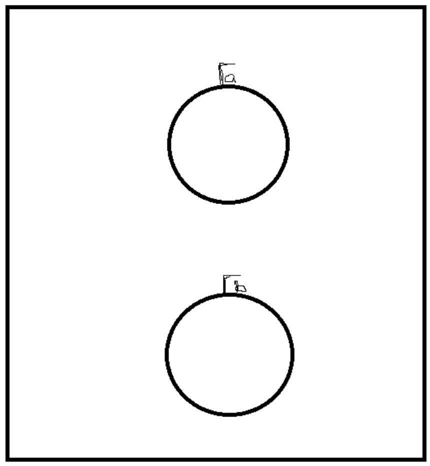

Figure 2: NSCs Γ_a and Γ_b on a triple torus

We now discuss some of the ways Γ_a and Γ_b can pass through given closed disks D_a and D_b. Let $L_a = \partial D_a$ and $L_b = \partial D_b$. Suppose $\Gamma_a \cap D_a$ has finitely many components including but not limited to Γ_a^1, Γ_a^2, Γ_a^3, Γ_a^4 and $\Gamma_b \cap D_b$ has finitely many components Γ_b^1, Γ_b^2, Γ_b^3, Γ_b^4 with;

i. Each $\Gamma_a^i \cap L_a$ has at most 2 components, $1 \leq i \leq 4$ and $\Gamma_b^j \cap L_b$ has at most 2 components $1 \leq j \leq 4$

ii. There is an arc P_1 in D_a with ends on L_a and $P_1^o \subset D_a^o$ such that $P_1 \cap \Gamma_a = x_1, x_2, x_3, x_4$ where x_1, x_2, x_3, x_4 are in that order along P_1. Each x_i belongs to Γ_a^i and Γ_a^2 and Γ_a^3 cross (not just intersect) P_1 at x_2 and x_3 respectively.

    Similarly there is an arc P_2 in D_b with ends on L_b and $P_2^o \subset D_b$ such that $P_2 \cap \Gamma_b = y_1, y_2, y_3, y_4$ where y_1, y_2, y_3, y_4 are in that order along P_2. Each y_i belongs to Γ_b^j and Γ_b^2 and Γ_b^3 cross P_2 at y_2 and y_3 respectively.

Assume that $x_1 P_1 x_2 \subset A_0$ and $y_1 P_2 y_2 \subset B_0$. For each i, let ia denote the first component of $\Gamma_a^i \cap L_a$ following Γ_a^i along Γ_a and ib the last. And for each j let jd denote the first component of $\Gamma_b^j \cap L_b$ following Γ_b^j along Γ_b and jf the last.

By the fact that Γ_a is separating and using orientations of A_0 and B_0 the components $1a$, $2b$, $3a$, $4b$, $4a$, $3b$, $2a$, $1b$ occur in that clockwise order along L_a (so $2a \neq 2b, 3a \neq 3b$, but possibly $1a = 1b$ or $4a = 4b$). Similarly by the fact that Γ_b is separating and using the orientations of B_0 and C_0 the components $1d$, $2e$, $3d$, $4e$, $4d$, $3e$, $2d$, $1e$ occur in that clockwise order along L_b (so $2e \neq 2d, 3d \neq 3e$ but possibly $1d = 1e$, or $4e = 4d$)

There are six possible cyclic orders in which components Γ_a^1, Γ_a^2, Γ_a^3, Γ_a^4 can occur along Γ_a and six possible cyclic orders in which components Γ_b^1, Γ_b^2, Γ_b^3, Γ_b^4 can occur along Γ_b. In each case they occur in pairs which are equivalent up to reversal of Γ_a and Γ_b respectively. If we know that Γ_a intersects the (closures of the) components of $D_a \backslash \Gamma_a$ essentially, then for some of these orders we can place restrictions on where additional components of $\Gamma_a \cap D_a$ can be. Similarly if we know that Γ_b intersects the (closures of the) components of $D_b \backslash \Gamma_b$ essentially, then for some of these orders we can place restrictions on where additional components of $\Gamma_b \cap D_b$ can be.

The following lemma shows possible orders of arrangements of segments of $\Gamma_a \cap D_a$ and $\Gamma_b \cap D_b$ on a triple torus.

Lemma 2.1.1: Suppose Γ_a and Γ_b are two non-contractible separating circles of S_3 with disks D_a and D_b and components Γ_a^1, Γ_a^2, Γ_a^3, Γ_a^4 of $\Gamma_a \cap D_a$ and components Γ_b^1, Γ_b^2, Γ_b^3, Γ_b^4 of $\Gamma_b \cap D_b$ as described above. Suppose further that Γ_a intersects the closure of every component of $D_a \backslash \Gamma_a$ essentially (in A_0 or B_0 as appropriate) and that Γ_b intersects the closure of every component of $D_b \backslash \Gamma_b$ essentially (in B_0 or C_0 as appropriate).

1. If the components occur along Γ_a in the order (1432) then $\Gamma_a \cap (2aL_a1b)^o = \Gamma_a \cap (4aL_a3b)^o = \emptyset$ and if the components occur along Γ_b in the order (1432) then $\Gamma_b \cap (2dL_b1e)^o = \Gamma_b \cap (4dL_b3e)^o = \emptyset$
2. If the components occur along Γ_a in the order (1342) then $\Gamma_a \cap (2aL_a1b)^o = \Gamma_a \cap (3aL_a4b)^o = \emptyset$ and if the components occur along Γ_b in the order (1342) then $\Gamma_b \cap (2dL_b1e)^o = \Gamma_b \cap (3dL_b4e)^o = \emptyset$

Proof: Suppose in Case 1 that $\Gamma_a \cap (2aL_a1b)^o \neq \emptyset$. Let Γ_a^5 (or for short 5) be the component of $\Gamma \cap (2aL_a1b)^o$ closest to 1b on L_a. By lemma 2.2 (iii), $5 \subset (2b\Gamma_a1a)^o$. But then $5L_a1b$ violates $C_aS_a(1aL_a2b, 3aL_a4b)(i)$ (i,e violates Lemma 2.1(i)). Here it is necessary if $\Gamma_a^1 = 1a = 1b$ is a single point. Thus, $\Gamma_a \cap (2aL_a1b)^o = \emptyset$. Similarly suppose that $\Gamma_b \cap (2dL_b1e)^o \neq \emptyset$. Let Γ_b^5 be the component of $\Gamma_b \cap (2dL_b1e)^o$ closest to 1e on L_b. By lemma 2.2 (iii), $\Gamma_b^5 \subset (2e\Gamma_b1d)^o$. But then $5L1e$ violates $C_bS_b(1dL_b2e, 3dL_b4e)(ii)$(i.e violates Lemma 2.1 (ii)). Here it is necessary if $\Gamma_b^1 = 1d = 1e$ is a single point. Thus $\Gamma_b \cap (2dL_b1e)^o = \emptyset$.

Similarly suppose in Case 2 that $\Gamma_a \cap (2aL_a1b) \neq \emptyset$. Let Γ_a^5 be the component of $\Gamma_a \cap (2aL_a1b)^o$ closest to 1b on L_a. By lemma 2.2 (iii), $5 \subset (2b\Gamma_a1a)^o$. But then $5L_a1b$ violates $C_aS_a(1aL_a2b, 4aL_a3b)(i)$(i.e violates Lemma 2.1 (i)). (Note: We assume that essential arcs can be shifted slightly if necessary to apply the Cylinder-Strip Lemma). Here it is necessary if $\Gamma_a^1 = 1a = 1b$ is a single point. Thus, $\Gamma_a \cap (2aL_a1b)^o = \emptyset$. On the other hand suppose that $\Gamma_b \cap (2dL_b1e)^o \neq \emptyset$. Let Γ_b^5 be the component of $\Gamma_b \cap (2dL_b1e)^o$ closest to 1e on L_b. By lemma 2.2 (iii), $\Gamma_b^5 \subset (2e\Gamma_b1d)^o$. But then $5L1e$ violates $C_bS_b(1dL_b2e, 4dL_b3e)(ii)$(i.e violates Lemma 2.1 (ii)). Here it is necessary if $\Gamma_b^1 = 1d = 1e$ is a single point. Thus $\Gamma_b \cap (2dL_b1e)^o = \emptyset$. The rest of the proof is similar.

2.2 Noncontractible Separating Circles (NSCs)

Next we discuss a method of constructing new NSCs from old ones on general surfaces in theorem 2.2.1 and Corollary 2.2.2 will show how this applies to a triple torus.

Theorem 2.2.1: Let Σ be a surface with two oriented NSCs Γ_a and Γ_b separating the surface into 3 closed components A_0, B_0 and C_0. Suppose there are sections Γ_a^1, Γ_a^2, Γ_a^3, Γ_a^4 of Γ_a in that order along Γ_a, and sections Γ_b^1, Γ_b^2, Γ_b^3, Γ_b^4 of Γ_b in that order along Γ_b. Suppose further that there are arcs P_{12}, P_{34} in A_0, Q_{23}, Q_{41} in B_0, also, Q_{12}, Q_{34} in B_0 and R_{23}, R_{41} in C_0, such that;

(i) Γ_a^1, Γ_a^2, Γ_a^3, Γ_a^4 are disjoint, so are Γ_b^1, Γ_b^2, Γ_b^3, Γ_b^4

(ii) P_{12}^o, P_{34}^o, Q_{23}^o, Q_{41}^o are disjoint from Γ_a and Q_{12}^o, Q_{34}^o, R_{23}^o, R_{41}^o are disjoint from Γ_b

(iii) P_{12} has ends a_1, a_2; P_{34} has ends a_3, a_4; Q_{23} has ends b_2, b_3; Q_{41} has ends b_4, b_1 where a_i, b_i are two ends of each Γ_a^i (not necessarily in order along Γ_a); Q_{12} has ends c_1, c_2; Q_{34} has ends c_3, c_4; R_{23} has ends d_2, d_3 and R_{41} has ends d_4, d_1; where c_j, b_j are two ends of each Γ_b^j (not necessarily in order along Γ_b).

(iv) $P_{12} \cup P_{34}$ separates A_0 into a component A_1 with boundary $P_{12} \cup a_2\Gamma_a a_3 \cup P_{34} \cup a_4\Gamma_a a_1$ (one circle) and a component A_2 with boundary $(a_1\Gamma_a a_2 \cup P_{12}) \cup (a_3\Gamma_a a_4 \cup P_{34})$ (two circles). $Q_{23} \cup Q_{41}$ similarly separates B_0 into a component B_1 with boundary $Q_{23} \cup b_3\Gamma_a b_4 \cup Q_{41} \cup b_1\Gamma_a b_2$ (one circle) and a component B_2 with boundary $(b_2\Gamma_a b_3 \cup Q_{23}) \cup (b_4\Gamma_a b_1 \cup Q_{41})$ (two circles). In the same way, $Q_{12} \cup Q_{34}$ separates B_0 into a component B_3 with boundary $Q_{12} \cup c_2\Gamma_b c_3 \cup Q_{34} \cup c_4\Gamma_b c_1$ (one circle) and a component B_4 with boundary $(c_1\Gamma_b c_2 \cup Q_{12} \cup (c_3\Gamma_b c_4 \cup Q_{34})$ (two circles), while $R_{23} \cup R_{41}$ similarly separates C_0 into a component C_1 with boundary $R_{23} \cup d_3\Gamma_b d_4 \cup R_{41} \cup d_1\Gamma_b d_2$ (one circle) and a component C_2 with boundary $(d_2\Gamma_b d_3 \cup R_{23}) \cup (d_4\Gamma_b d_1 \cup R_{41})$ (two circles). Then;

$\Gamma' = \Gamma_a^1 \cup P_{12} \cup \Gamma_a^2 \cup Q_{23} \cup \Gamma_a^3 \cup P_{34} \cup \Gamma_a^4 \cup Q_{41}$ and

$\Gamma'' = \Gamma_b^1 \cup Q_{12} \cup \Gamma_b^2 \cup R_{23} \cup \Gamma_b^3 \cup Q_{34} \cup \Gamma_b^4 \cup R_{41}$ are also Non-contractible separating circles in Σ^2, separating $A_1 \cup B_2$ from $A_2 \cup B_1$ and $B_3 \cup C_2$ from $B_4 \cup C_1$ respectively.

Proof. We observe from the conditions of the theorem that Γ' separates $A_1 \cup B_2$ from $A_2 \cup B_1$ and that Γ'' separates $B_3 \cup C_2$ from $B_4 \cup C_1$. It is enough to show that both Γ' and Γ'' are Non-contractible or equivalently that none of $A_1 \cup B_2$, $A_2 \cup B_1$,$B_3 \cup C_2$ and $B_4 \cup C_1$ is homeomorphic to a disk.

Since A_1 has one boundary circle, it is homeomorphic to a disk with handles and or crosscaps attached. Since B_2 has two boundary circles, it is homeomorphic to a cylinder with handles and or crosscaps attached. If A_1 is just a disk and B_2 just a cylinder, the way they are attached along the segments of Γ_a between Γ_a^2 and Γ_a^3 and between Γ_a^4 and Γ_a^1 means that the result would be homeomorphic to a punctured torus. More generally, the result is homeomorhic to a punctured torus with handles and or crosscaps added, which is not a disk. Therefore $A_1 \cup B_2$ is not a disk. Similarly, it can be shown that $A_2 \cup B_1$, $B_3 \cup C_2$ and $B_4 \cup C_1$ are not disks, then

$\Gamma' = \Gamma_a^1 \cup P_{12} \cup \Gamma_a^2 \cup Q_{23} \cup \Gamma_a^3 \cup P_{34} \cup \Gamma_a^4 \cup Q_{41}$

is Non-contractible and

$\Gamma'' = \Gamma_b^1 \cup Q_{12} \cup \Gamma_b^2 \cup R_{23} \cup \Gamma_b^3 \cup Q_{34} \cup \Gamma_b^4 \cup R_{41}$ is also Non-contractible.

We now apply this to the triple torus, with weaker versions of conditions (ii) and (iv), and stating condition (iv) in a way that is specific to the triple torus.

Corollary 2.2.2. Suppose Γ_a and Γ_b are two Non-contractible separating circles in the triple torus S_3, separating S_3 into three (closed) punctured tori A_0, B_0 and C_0. Suppose there are sections Γ_a^1, Γ_a^2, Γ_a^3, Γ_a^4 of Γ_a in that order along Γ_a, and sections Γ_b^1, Γ_b^2, Γ_b^3, Γ_b^4 of Γ_b in that order along Γ_b, arcs P_{12}, P_{34} in A_0; Q_{23}, Q_{41} in B_0 ; Q_{12}, Q_{34} in B_0 and R_{23}, R_{41} in C_0 such that;

(i) Γ_a^1, Γ_a^2, Γ_a^3, Γ_a^4 are disjoint, so are Γ_b^1, Γ_b^2, Γ_b^3, Γ_b^4

(ii) P_{12}^o, P_{34}^o, Q_{23}^o, Q_{41}^o are disjoint from Γ_a and Q_{12}^o, Q_{34}^o, R_{23}^o, R_{41}^o are disjoint from Γ_b

(iii) P_{12} has ends a_1, a_2; P_{34} has ends a_3, a_4; Q_{23} has ends b_2, b_3 and Q_{41} has ends b_4, b_1 where a_i, b_i are two ends of each Γ_a^i (not necessarily in order along Γ_a) and Q_{12} has ends c_1, c_2, Q_{34} has ends c_3, c_4, R_{23} has ends d_2, d_3 and R_{41} has ends d_4, d_1 where c_j, b_j are two ends of each Γ_b^j (not necessarily in order along Γ_b).

(iv) P_{12}, P_{34} are homotopic with endpoints fixed in A_0 to a pair of parallel essential arcs and Q_{23}, Q_{41} are homotopic with endpoints fixed in B_0 to a pair of parallel essential arcs. Similarly Q_{12}, Q_{34} are homotopic with endpoints fixed in B_0 to a pair of parallel essential arcs and R_{23}, R_{41} are homotopic with endpoints fixed in C_0 to a pair of parallel essential arcs. Then;

$\Gamma' = \Gamma_a^1 \cup P_{12} \cup \Gamma_a^2 \cup Q_{23} \cup \Gamma_a^3 \cup P_{34} \cup \Gamma_a^4 \cup Q_{41}$ and

$\Gamma'' = \Gamma_b^1 \cup Q_{12} \cup \Gamma_b^2 \cup R_{23} \cup \Gamma_b^3 \cup Q_{34} \cup \Gamma_b^4 \cup R_{41}$ are also Non-contractible separating circles in S_3.

Proof: Conditions (ii) and (iv) mean that by shifting Γ_a slightly, we can make P_{12}, P_{34} and Q_{23}, Q_{41} into pairs of parallel essential arcs. Similarly, by shifting Γ_b slightly we can make Q_{12}, Q_{34} and R_{23}, R_{41} into pairs of parallel essential arcs. Then each of the (slightly shifted) punctured tori is separated into a cylinder and a strip in each case. Now applying Theorem 2.2.1 completes the proof.

Sets of arcs P_{12}, P_{34}, Q_{23}, Q_{41} and Q_{12}, Q_{34}, R_{23}, R_{41} satisfying Corollary 2.2.2 are called orthogonal arrangements of parallel arcs, or OP for short. They are referred to as $OP(P_{12}, P_{34}; Q_{23}, Q_{41})$ and $OP(Q_{12}, Q_{34}; R_{23}, R_{41})$. The arcs P_{12}, P_{34}, Q_{23}, Q_{41} are not required to have interiors disjoint from Γ_a, just from sections Γ_a^1, Γ_a^2, Γ_a^3, Γ_a^4. Similarly, the arcs Q_{12}, Q_{34}, R_{23}, R_{41} are not required to have interiors disjoint from Γ_b, but just from sections Γ_b^1, Γ_b^2, Γ_b^3, Γ_b^4. The most common case of this is illustrated in figure 3. When forming an OP, the two hatched essential arcs joined by a hatched segment of Γ may be considered equivalent to the single dashed essential arc, as long as the hatched segment of Γ_a does not intersect $\Gamma_a^1, \Gamma_a^2, \Gamma_a^3, \Gamma_a^4$.

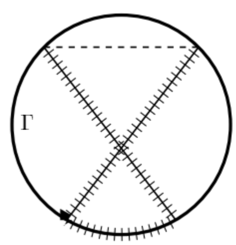

Figure 3: Equivalent to an essential arc, see [**2**].

2.3 Critical Embeddings

In proving the existence of NSCs, Ellingman in [**2**] used an approach of examining embeddings which are very close to having an NSC.

The following lemma justifies existence of NSCs in critical embeddings with critical edges.

Lemma 2.3.1 : Let Σ be a suitable surface (one with genus at least 2), and $k \geq 3$ (the representativity of an embedding on such a surface). Suppose there is a k-representative embedding in Σ that does not have an NSC. Then there exists a k-representative embedding Ψ of a simple 2-connected graph in Σ that does not have an NSC, with a face f containing non-adjacent vertices x, y so that when the edge xy is inserted across the face f, $\Psi^+ = \Psi \cup xy$ has an NSC. We call Ψ a critical embedding with critical edge xy, see [**2**].

Proof: Let Ψ_0 be a k-representative embedding of a graph G_0 in Σ with no NSC. Since $k \geq 3$ and since multiple edges bounding a disk can be reduced to a single edge without affecting existence of an NSC, we may assume that G_0 is simple. Moreover, by reducing it to essential 2-component, we may assume that G_0 is 2-connected.

Ellingman N. & Zha X in [**2**] defined an augmentation of an embedding to be either;

(i) The addition of an edge across a face between two nonadjacent vertices on that face or,

(ii) If every face is a triangle as in triangulation (bounded by a 3-cycle), then in some face (uvw) subdivide one edge uv with a new vertex x and then add the edge ux

Neither (i) nor (ii) decreases the representativity. In a sequence of augmentations, any augmentation following one of type (ii) must be of type (i).

If we apply a sequence of augmentations to Ψ_0, each embedding is k-representative with a graph that is simple and 2-connected. Moreover, by applying a sequence of augmentations to Ψ_0 we can increase its representativity arbitrarily. First, we complete Ψ_0 to a triangulation using type (i) augmentations. It is well known that in a triangulation, representativity equals the length of a shortest noncontractible cycle.

Given an edge $e = vw$ on a shortest noncontractible cycle, belonging to two triangles (uvw) and (twv), we can apply four augmentations of type (ii), (i), (ii), (i) with the effect of deleting vw, adding two new vertices x_1, x_2, and adding paths vx_1w, vx_2w, ux_1x_2t. This destroys all shortest noncontractible cycles through e without creating any new shortest noncontractible cycles. After destroying all shortest noncontractible cycles in this way, the representativity must increase by at least one, then we can repeat the process. The essence of repeating the process is to raise representativity to 6 at which point Zha & Zhao in [**6**] proved that an NSC exists.

Therefore, it is possible to apply a sequence of augmentations to Ψ_0 to raise the representativity to at least 6, at which point an NSC exist. Let Ψ be an embedding in the sequence before the first augmentation that creates an NSC. Type (ii) augmentations cannot create an NSC if one does not already exist. So the augmentations is of type (i) and the result follows.

2.4 The Main Result

In this section, it is shown that every embedding on the triple torus with representativity at least 4 contains two NSCs. We begin with a standard result on 4-representative graphs.

Definition 2.4.1: Equivalence of embeddings: Two embeddings $\Psi_a : G \longrightarrow \Sigma$ and $\Psi_b : G \longrightarrow \Sigma$ are equivalent, denoted by $\Psi_a \backsimeq \Psi_b$ if there exists a graph automorphism $\alpha : G \longrightarrow G$ and an orientation preserving surface homeomorphism $\eta : \Sigma \longrightarrow \Sigma$ for which the map compositions $\Psi_a \circ \eta : G \longrightarrow \Sigma$ and $\alpha \circ \Psi_b : G \longrightarrow \Sigma$ are identical , see [**4**].

The following lemma shows how two, 4-representative graph embeddings on the triple torus are equivalent.

Lemma 2.4.2: Let $\Psi_a : G \longrightarrow S_3$ and $\Psi_b : G \longrightarrow S_3$ be two embeddings of a graph G into the triple torus S_3. Suppose there exists an automorphism $\alpha : G \longrightarrow G$ between G and itself. Then the embeddings $\Psi_a : G \longrightarrow S_3$ and $\Psi_b : G \longrightarrow S_3$ are equivalent.

Proof: First we show that there is an orientation preserving surface homeomorphism $\eta : S_3 \longrightarrow S_3$. [Bernadi. O (2011)] argued that a map is a connected graph embedded on a surface, considered up to orientation preserving homeomorphism. This is obvious since a triple torus is always homeomorphic to itself. For easy identification of the preservation of the orientation, we consider the case where the homeomorphism η is an identity map.

Finally we show that the map compositions $\Psi_a \circ \eta : G \longrightarrow S_3$ and $\alpha \circ \Psi_b : G \longrightarrow S_3$ are identical. Let e be an arc with end points v, w on S_3, then $\eta(e) = e$ since η is identity map. Since Ψ_a is an embedding, $\Psi_a((u, v)) = e$. Similarly, let e be an edge of graph G with end points u, v, i.e, edge (u, v). Then since Ψ_b is an embedding, $\Psi_b((u, v)) = e$ where e is an arc on S_3. Applying η on e we have $\eta(e) = e$ since e is an identity map. Hence, the two embeddings Ψ_a and Ψ_b are equivalent.

The lemma below shows the existence of a disk D which contains the union of f and all faces that share at least one vertex with f, given that f is the face of an embedding which has representativity at least 3. The subsequent lemma shows how this applies to 4-representative embeddings on the triple torus.

Lemma 2.4.3: Let f be a face of an embedding $\Psi : G \to \Sigma$, where Σ is not a 2-sphere. Let k be an integer such that $\rho(\Psi, f) > 2k + 1$ and let $B_k(f)$ be the union of f and all faces that share at least one vertex with f. Then there is a disk $D_k(f) \subset \Sigma$

which contains $B_k(f)$ such that $\partial D_k(f) \subseteq \partial B_k(f)$, see [7].

Proof: Any contractible simple closed curve $\gamma \in \Sigma$ bounds a unique disc since Σ is not a 2-sphere. Denote this disc by $int(\gamma)$. Clearly, any disc containing $B_k(f)$ must contain the disk $int(\gamma)$ for any simple closed curve γ in $B_k(f)$. Let

$D_k = B_k(f) \cup \{int(\gamma) \mid \gamma \text{ is a simple closed curve in } B_k(f)\}.$

Each closed curve γ in D_k is homotopic to some closed curve contained in $B_k(f)$ since any part of γ in $int(\gamma)$ can be moved by homotopy to the boundary of $int(\gamma)$, which is contained in $B_k(f)$. D_k is simply connected and it is also connected by construction and $\partial D_k \subseteq \partial B_k(f)$. Since the only simply connected compact surfaces are the 2-sphere and the closed disk, it suffices to show that D_k is a 2-manifold with boundary. By construction it follows that D_k is closed. Moreover D_k is a union of closed faces. Therefore, a singularity can only appear at the vertex of the embedded graph. But by the following reason a true singularity is excluded. If g and h are faces in $B_k(f)$ meeting at a vertex x, let γ be a closed curve starting at a point in $int(f)$ leading to g going through x to h, and returning to f, such that $\mid z \in S^1 \mid \gamma(z) \in \Psi(G) \mid \leq 2k+1$. Since $\rho(f) > 2k+1$, γ bounds a disk in D_k. Consequently all the faces at x which lie between g and h (one or the other side) also lie in D_k.

Lemma 2.4.4: Let f_a and f_b be faces of two 4-representative embeddings Ψ_a and Ψ_b which are equivalent on the triple torus, and let F_a be the union of f_a and all faces that share at least one vertex with f_a. Similarly let F_b be the union of f_b and all faces that share at least one vertex with f_b.

(i) The face f_a is a disk D_a^1 with boundary cycle L_a^1, and there is a disk $D_a^2 \supset D_a^1$ with boundary cycle L_a^2, such that $F \subseteq D_a^2$ and $L_a^2 = \partial D_a^2 \subseteq \partial F_a$. Similarly, the face f_b is a disk D_b^1 with boundary cycle L_b^1, and there is a disk $D_b^2 \supset D_b^1$ with boundary cycle L_b^2, such that $F \subseteq D_b^2$ and $L_b^2 = \partial D_b^2 \subseteq \partial F_b$

(ii) Any path P_a in D_a^2 with both ends on L_a^2 must be a segment of L_a^2 or must contain a vertex of L_a^1 and any path P_b in D_b^2 with both ends on L_b^2 must be a segment of L_b^2 or must contain a vertex of L_b^1

Proof.

i (i) is a special case of Lemma 2.4.3.

ii If (ii) fails, there would be a path P_a in D_a^2 internally disjoint from L_a^2 joining two vertices of L_a^2 and not intersecting L_a^1. Labeling the ends a, b of P_a appropriately, $P_a \cup bL_a^2a$ would separate $(aL_a^2b)^o$ from L_a^1. But this contradicts the fact that since $L_a^2 \subseteq \partial F_a$, every point of L_a^2 has an arc joining it to L_a^1 that does not intersect the graph except at its endpoints. Similarly, there would be a path P_b in D_b^2

internally disjoint from L_b^2 joining two vertices of L_a^2 and not intersecting L_b^1. Labeling the ends d, e of P_b appropriately, $P_b \cup dL_b^2 e$ would separate $(dL_b^2 e)^o$ from L_b^1. But this contradicts the fact that since $L_b^2 \subseteq \partial F_b$, every point of L_b^2 has an arc joining it to L_b^1 that does not intersect the graph except at its endpoints.

We now present the main theorem.

Theorem 2.4.5: Every 4-representative embedding on the triple torus contains two NSCs which splits the triple torus into 3 connected components.

Proof: Suppose the theorem is false. By Lemma 2.3.1, there are critical 4-representative embeddings Ψ_a and Ψ_b (of simple 2-connected graphs) with no NSC, while $\Psi_a^+ = \Psi_a \cup vw$ and $\Psi_b^+ = \Psi_b \cup xy$ have NSCs. Suppose that vw is added across the face f_a and xy across the face f_b. Let D_a^1, D_a^2, L_a^1, L_a^2 and let D_b^1, D_b^2, L_b^1, L_b^2 be as provided by Lemma 2.4.4 for both f_a and f_b and let $L_a = L_a^1 \cup L_a^2$ and $L_b = L_b^1 \cup L_b^2$.

Every NSC in Ψ_a^+ must contain the edge vw and every NSC in Ψ_b^+ must contain the edge xy. Of all NSCs in Ψ_a^+, let Γ_a be the one that minimises $\| \Gamma_a \cap D_a^2 \|$ (the number of components in $\Gamma_a \cap D_a^2$) and subject to this also minimises $\| \Gamma_a \cap D_a^1 \|$. Similarly let Γ_b be the one that minimises $\| \Gamma_b \cap D_b^2 \|$ (the number of components in $\Gamma_b \cap D_b^2$) and subject to this also minimises $\| \Gamma_b \cap D_b^1 \|$.

Then each component of $\Gamma_a \cap D_a^2$ contains at most one component of $\Gamma_a \cap D_a^1$ (and using lemma 2.4.4 (ii)), at most 2 components of $\Gamma_a \cap L_a^2$. By the same argument, each component of $\Gamma_b \cap D_b^2$ contains at most one component of $\Gamma_b \cap D_b^1$ (and using lemma 2.4.4 (ii)), at most 2 components of $\Gamma_b \cap L_b^2$. We often abbreviate Γ_a^i to i. (In later parts of the proof we also use components of Γ_a', abbreviated i', where $i = 1, 2, ...$ and components of Γ_b', abbreviated j', where $j = 1, 2, ...$). We represent subsegments of component i as ij where j is a letter, e.g $3a$ is a subsegment of $3 = \Gamma_a^3$.

Since Ψ_a and Ψ_b are equivalent by lemma 2.4.2, all operations on Γ_a applies similarly on Γ_b. We now focus on Γ_a and conclude to operations of Γ_b by equivalence of the embeddings.

The minimality assumption further guarantees that any arc in D_a^2 that joins different components Γ_a^i, Γ_a^j of $\Gamma \cap D_a^2$ and is otherwise disjoint from Γ_a is essential. If it were not essential then we could replace one of the segments $i\Gamma_a j$ or $j\Gamma_a i$ with a section of L_a^2, reducing $\| \Gamma_a \cap D_a^2 \|$. This is true even if the segment of L_a^2 we wish to use, intersects other components of $\Gamma_a^2 \cap D_a^2$, because those other components must also be part of the segment of Γ_a we are replacing.

Let Γ_a^3 be the component of $\Gamma_a \cap D_a^2$ that contains vw. Then $\Gamma_a^3 \cap L_a$ has four components which we name $3a$, $3b$, $3c$, $3d$ in order along Γ_a with $3a, 3d \subset L_a^2$ and $3b, 3c \subset L_a^1$. We may assume that $v \in 3b$ and $w \in 3c$. For ease of description, we assume that D_a^2 is drawn as a circular disk and Γ_a^3 passes downwards through D_a^2, with $3a$ containing its top point and $3d$ its bottom point. Other than Γ_a^3, no other component of $\Gamma_a \cap D_a^2$ contains more than one component of $\Gamma_a \cap L_a^1$.

In fact, any other component Γ_a^i of $\Gamma_a \cap D_a^2$ is one of two types. If $\| \Gamma_a^i \cap D_a^1 \| = 1$, i is a segment of L_a^2. Otherwise, $\| \Gamma_a^i \cap L_a \| = 3$ and i includes two segments ia, ic of L_a^2 and one segment ib of L_a^1 with ia, ib, ic in that order along Γ_a. Since Ψ_a is critical, Γ_a intersects both $(3bL_a^1 3c)^o$ and $(3cL_a^1 3b)^o$, otherwise we could reroute Γ_a to avoid (the interior of) $3b\Gamma_a 3c = vw$. Moreover, each component of $\Gamma_a \cap D_a^2$ which intersects $(3cL_a^1 3b)^o$ cannot be rerouted via $3dL_a^2 3a$ or we could reduce $\| \Gamma_a \cap D_a^1 \|$.

Let Γ_a^2 denote any such component, with $2a, 2c \subset L_a^2$ and $2b \subset L_a^1$. Note that Γ_a^2 passes upwards through D_a^2, so that the half of f_a to the right of Γ_a^3 is also to the right of Γ_a^2. Since Γ_a^2 cannot be rerouted there is at least one component of $\Gamma_a \cap D_a^2$ that intersects $(2aL_a^2 2c)^o$. Let Γ_a^1 denote any such component, which must be a segment of L_a^2 and passes downwards through D_a^2. In similar way, we can find Γ_a^4 passing upwards through D_a^2, intersecting $(3aL_a^2 2c)^o$ at $4a$ and $4c$ and intersecting $(3bL_a^1 3c)^o$ at $4b$. Then we must also have $\Gamma_a^5 = 5$ passing downwards through D_a^2 and contained in $(4cL_a^2 4a)^o$. In general, it is not known whether a given component of $\Gamma_a \cap L_a$ is trivial(single vertex) or not. The above description can be well represented by figure 4.

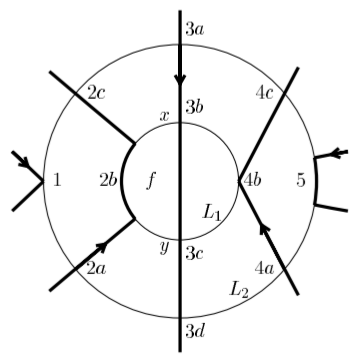

Figure 4: Segments of Γ in D_2 , see [**2**].

Let A_0 denote the part of S_3 to the left of Γ_a, and B_0 the part to the right; Also B_0 is the part of S_3 to the left of Γ_b and C_0 to the right. A_0, B_0 and C_0 are punctured tori. For $i \geq 1$, let A_i denote the unique component of $A_0 \cap D_a^2$ to which Γ_a^i belongs; define B_i similarly. (If $\Gamma_a^i \subset L_a^2$, one of A_i or B_i will be just Γ_a^i itself). We know that $A_1 = A_2$, $B_2 = B_3$, $A_3 = A_4$ and $B_4 = B_5$.

We will frequently use the orthogonal arrangements of parallel paths to construct a new NSC Γ_a' in Ψ_a^+. In notation, $OP(P, P'; Q, Q')$, P, P' will be paths in A_0 and Q, Q' those in B_0. There are two common ways in which this provides a contradiction.

First, Γ'_a may avoid the edge vw, and so be an NSC for Ψ_a; we indicate this by $AOP(P, P'; Q, Q')$. Second, $\Gamma'_a \cap D^2_a$ may have fewer components than $\Gamma_a \cap D^2_a$; we indicate this by $COP(P, P'; Q, Q')$.

Suppose P, P', Q, Q' all lie in D^2_a. When we form Γ'_a from Γ_a we delete the interiors of four nontrivial segments of Γ_a, say S_1, S_2, S_3, S_4 and then add the interiors of P, P', Q, Q'. Each end of S_j lies in some component of $\Gamma_a \cap D^2_a$. Suppose each S_j intersects s_j components of $\Gamma_a \cap D^2_a$, then $s_j \geq 1$. When we delete S^o_j, the number of components in D^2_a changes by $2 - s_j$. When we add P^o, P'^o, Q, Q'^o, the number of components in D^2_a changes by -4. Thus $\parallel \Gamma'_a \cap D^2_a \parallel = \parallel \Gamma_a \cap D^2_a \parallel$ $+4 - s_1 - s_2 - s_3 - s_4$.

The above analysis is valid even when the interiors of P, P', Q, Q' intersect components of $\Gamma_a \cap D^2_a$. If we do not have $s_1 = s_2 = s_3 = s_4 = 1$, then we have $COP(P, P'; Q, Q')$. In particular, let $OOP[i](P, P'; Q, Q')$ denote the situation in which some component i of $\Gamma_a \cap D^2_a$ contains an odd number of the eight endpoints of $P, P'; Q, Q'$ (counted with multiplicity). Then $s_j > 1$ for some j, so this is a special case of $COP(P, P'; Q, Q')$.

Now we break into cases according to the order of 1,2,3,4,5 along Γ_a. For any distinct components i_1, i_2,...,i_k of $\Gamma_a \cap D^2_a$, we say that Γ_a has $(i_1 i_2, ..., i_k)$ if i_1, i_2,...,i_k occur in that order along Γ_a.

(A) Suppose Γ_a has (1432). (Since we do not mention component 5, no assumption is made about its position). By lemma 2.1.1, $\Gamma_a \cap (2aL^2_a 1)^o = \Gamma_a \cap (4aL^2_a 3d)^o = \emptyset$. Note that by lemma 2.2, $\Gamma_a \cap (2cL^2_a 3a)^o \subset (3\Gamma_a 2)^o$.

For easy understanding of the subsequent arguments in the proof, figure 5 shows how we construct new NSCs using corollary 2.2.2 in the first two cases here. The solid chords are essential paths in A_0, the dashed chords are essential paths in B_0, and the edges of Γ used by the new NSC are hatched.

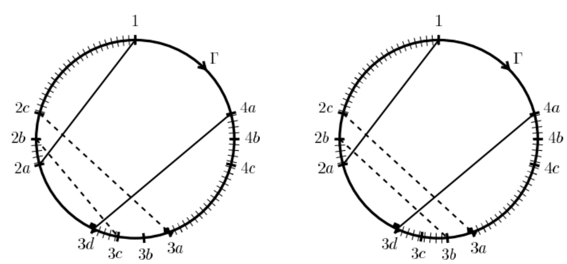

Figure 5: Construction of new NSCs in the first 2 cases of A. Case 1 (left), case 2 (right) , see [**2**].

First, suppose that $\Gamma_a \cap (3cL^1_a 2b)^o = \emptyset$. Let $P = 2c(\partial B_3)3a$ (P is just $2cL^2_a 3a$ if $\Gamma_a \cap (2cL^2_a 3a)^o = \emptyset$). Since $\Gamma_a \cap (2cL^2_a 3a)^o$ is contained in $(3\Gamma_a 2)^o$, we have $P^o \cap \Gamma_a \subset$

$(3\Gamma_a 2)^o$. Now we have $AOP(2aL_a^2 1, 4aL_a^2 3d; 3cL_a^1 2b, P)$. This is illustrated on the left of figure 5. Note that $(3\Gamma 2)^o$ is not hatched, showing that this part of Γ may be used by P if necessary.

Second, suppose that $\Gamma_a \cap (2bL_a^1 3b)^o = \emptyset$. Since $\Gamma_a^1 \cap (2cL_a^2 3a)^o \subset (3\Gamma_a 2)^o$, we may have $OOP[1](2aL_a^2 1, 4aL_a^2 3d; 2cL_a^2 3a, 2bL_a^1 3b)$. This is illustrated on the right of figure 5 Though these two cases appear similar, they are different.

Finally, we may suppose that there exists $\Gamma_a^6 = 6$ that intersects $(2bL_a^1 3b)^o$ and $\Gamma_a^7 = 7$ that intersects $(3cL_a^1 2b)^o$. By lemma 2.2, 2,3,6 and 7 are the only components of $\Gamma_a \cap D_a^2$ intersecting B_3, and Γ_a has (3627). If Γ_a has (271) then we have $OOP[1](2aL_a^2 1, 4aL_a^2 3d; 2bL_a^1 6b, 3cL_a^1 7b)$. If Γ_a has (173) then we have $OOP[1](2aL_a^2 1, 4aL_a^2 3d; 2bL_a^2 6b, 7bL_a^1 2b)$. By symmetry we may also exclude the cases where Γ_a has (1234), (5234), or (5432).

(B) Suppose Γ_a has (1342). By Lemma 2.1.1, $\Gamma_a \cap (2aL_a^2 1)^o = \Gamma_a \cap (3aL_a^2 4c)^o = \emptyset$. Suppose that $\Gamma_a \cap (2cL_a^2 3a)^o = \emptyset$, then we have $OOP[1](2aL_a^2 1, 3aL_a^2 4c; 2cL_a^2 3a, 2bL_a^1 3b)$. Therefore, we may assume $\Gamma_a \cap (2cL_a^2 3a) \neq \emptyset$, and similarly $\Gamma_a \cap (3dL_a^2 2a)^o \neq \emptyset$. Let $\Gamma_a^6 = 6$ intersect $(2cL_a^2 3a)^o$, and $\Gamma_a^7 = 7$ intersect $(3dL_a^2 2a)^o$. By lemma 2.2, 2,3,6,7 are the only components of $\Gamma_a \cap D_a^2$ intersecting B_3 and Γ_a has (3627). Then we have $OOP[1](2aL_a^2 1, 3aL_a^2 4c; 2cL_a^2 6, 3dL_a^2 7)$. Figure 6 shows how new NSCs are constructed in this case.

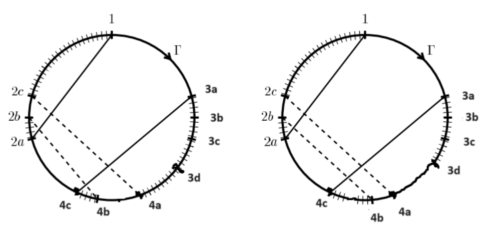

Figure 6: Construction of new NSCs in the first 2 cases of B. Case 1 (left), case 2 (right)

By symmetry, we may also exclude the cases where Γ_a has (1243), (5324) or (5423).

Now we know that Γ_a must have either (1324) or (1423), and either (5243) or (5342). So the overall order must be (13524) or (14253). These cases are symmetric, so let us assume the order is (14253). Given this order, there is a symmetry that reverses Γ_a and swaps A_0 and B_0. For our standard picture of D_a^2, this amounts to rotating D_a^2 by 180^o and reversing Γ_a.

Consider the components of $\Gamma_a \cap D_a^2$ that intersect $(2aL_a^2 2c)^o$. Each such component lies in $3\Gamma_a 4)^o$, otherwise we could choose that component as 1 and have case (A) or

(B). Let 1 be the first and $1'$ the last such component along $3\Gamma_a 4$. (Possibly $1 = 1'$). By Lemma 2.2 there are at most three such components, and 1 is the first and $1'$ the last, along $2aL_a^2 2c$. Thus, $\Gamma_a \cap (2aL_a^2 1)^o = \Gamma(1'L_a^2 2c)^o = \emptyset$. If there are three distinct components 1, $1^\star$, $1'$ in order along Γ_a, then $3aL_a^2 4c$ violates $CS(1L_a^2 1^\star, 1'L_a^2 2c)(i)$ in A_0. Therefore, there are at most two such components. Similarly, at most two components of $\Gamma_a \cap D_a^2$ intersects $(4cL_a^2 4a)^o$, they lie in $(2\Gamma_a 3)^o$ and if 5 is the first and $5'$ the last along $2\Gamma_a 3$ (possibly 5=5′), then $\Gamma_a \cap (4cL_a^2 5')^o = \Gamma_a \cap (5L_a^2 4a)^o = \emptyset$.

If $\Gamma_a \cap (2cL_a^2 3a)^o \neq \emptyset$, we denote the component of $\Gamma_a \cap D_a^2$ closest to 3a by 6, and that closest to 2c by $6'$ (possibly 6=6′). If $\Gamma_a \cap (4aL_a^2 3d)^o \neq \emptyset$, we denote the component of $\Gamma_a \cap D_a^2$ closest to 2a by 7 and that closest to 3d by $7'$ (possibly 7=7′). By Lemma 2.2, Γ_a has $(366'277')$ suitably modified to identify components that are the same and delete components that do not exist. Similarly, if $\Gamma_a \cap (3aL_a^2 4c)^o \neq \emptyset$, we denote the component of $\Gamma_a \cap D_a^2$ closest to 3a by 8 and that closest to 4c by $8'$ (possibly 8=8′). If $\Gamma_a \cap (4aL_a^2 3d)^o \neq \emptyset$ we denote the component of $\Gamma_a \cap D_a^2$ closest to 4a by 9 and that closest to 3d by $9'$ (possibly 9=9′). By Lemma 2.2, Γ_a has $(388'499')$ suitably modified. Note that $6, 6', 7, 7', 8, 8', 9, 9'$ may or may not intersect L_a^1.

Claim 1. At least one of 7 and 8 exists

Proof. If not, we have the $OP(3bL_a^1 4b, 3aL_a^2 4c; 3cL_a^1 2b, 3dL_a^2 2a)$ which produces Γ_a' with $\| \Gamma_a' \cap D_a^2 \| = \| \Gamma_a \cap D_a^2 \|$ and $\| \Gamma_a' \cap D_a^1 \| = \| \Gamma_a \cap D_a^1 \| -2$ contradicting the minimallity of Γ_a.

Claim 2. At most one of 6 and 7 exists. By symmetry, at most one of 8 and 9 exists.

Proof. Suppose both 6 and 7 exists. By Lemma 2.2(i), 2, 3, 7 are the only components of $\Gamma_a \cap D_a^2$ intersecting B_3, and Γ_a has (3627). To avoid an arc (not necessarily path) from 4 to 5 in B_5 violating $CS(6L_a^2 3a, 7L_a^2 2a)(i)$ in B_0, Γ_a must have (275) when it has (374), and must have (573) when it has (462). So Γ_a has either (364275) or (346257).

Case (2.1) Suppose Γ_a has (364275). If 8 does not exist, let $P = 3aL_a^2 4c$ and $P' = 3bL_a^1 4b$; If 9 does not exist, let $P = 4aL_a^2 3d$ and $P' = 4bL_a^1 3c$. In either case we have $OOP[2](P, P'; 3dL_a^2 7, 2cL_a^2 6)$. Therefore, 8 and 9 exist. By lemma 1.3, 3,4,8,9 are the only components of $\Gamma_a \cap D_a^2$ intersecting A_3, and Γ_a has (3849).

To avoid an arc from 1 to 2 in A_1 violating $CS(3aL_a^2 8, 4aL_a^2 9)(i)$ in A_0, Γ_a must have (429) when it has (318), and must have (923) when it has (814). So Γ_a has either (318429) or (381492). If Γ_a has (318429) we have $OOP[2](8L_a^2 4c, 9L_a^2 3d; 6L_a^2 3a, 7L_a^2 2a)$. If Γ_a has (381492) we have $OOP[5](8L_a^2 4c, 9L_a^2 3d; 6L_a^2 3a, 5L_a^2 4a)$.

Case (2.2) Suppose Γ_a has (346257). If $\Gamma_a \cap (3aL_a^2 4c)^o = \emptyset$ let $P = 3aL_a^2 4c$ and $P' = 3bL_a^1 4b$; if $\Gamma_a \cap (4aL_a^2 3d)^o = \emptyset$ let $P = 4aL_a^2 3d$ and $P' = 4bL_a^2 3c$. In either case

we have
$OOP[5](P, P'; 3dL_a^2 7, 5L_a^2 4a)$.

Claim 3. If 7 exists, then $7 = 7'$ and Γ_a has (275). By symmetry, if 8 exists then $8 = 8'$ and Γ_a has ($1'$84).

Proof. We first show that Γ_a does not have (364). Suppose Γ_a has (364). Since at most one of 8 and 9 exists by Claim 2, we may take paths P, P' to be either $3aL_a^2 4c$, $3bL_a^1 4b$ or $4bL_a^1 3c$, $4aL_a^1 3d$. Then we have $OOP[5](P, P'; 3dL_a^2 7', 5L_a^2 4a)$.

If $7 \neq 7'$ then to avoid an arc from 4 to 5 violating $CS(3dL_a^2 7', 7L_a^2 2a)(i)$ in B_0, Γ_a must have $(2757'3)$ and hence $(57'3)$. Thus $7 = 7'$, and since Γ_a does not have $(57'3) = (573)$, it must have (275).

Claim 4. If 6 exists then $6 = 6'$ and Γ has (462). By symmetry, if 9 exists then $9 = 9'$ and Γ_a has (492).

Proof. We first show that Γ_a does not have (364). Suppose Γ_a has (364). Since at most one of 8 and 9 exists by Claim 2, we may take paths P, P' to be either $3aL_a^2 4c$, $3bL_a^2 4b$ or $4bL_a^1 3c$, $4aL_a^2 3d$. Then we have $OOP[5](P, P'; 6L_a^2 3a, 5L_a^2 4a)$.

If $6 \neq 6'$ then to avoid an arc from 4 to 5 violating $CS(2cL_a^2 6', 6L_a^2 3a)(i)$ in B_0, Γ_a must have $(3646'2)$ and hence (462).

Now from claim 1 we may assume without loss of generality that 7 exists. By claim 2, 6 does not exist, and at most one of 8 or 9 exists. If 8 exists then Γ_a has $(31'84275)$ by claim 3, and we get $OOP[1'](1'L_a^2 2c, 8L_a^2 4c; 5L_a^2 4a, 7L_a^2 2a)$. If 9 exists, then Γ_a has $(31'49275)$ by claim 4 and we get $OOP[1'](1'L_a^2 2c, 4aL_a^2 9; 5L_a^2 4a, 7L_a^2 2a)$. Therefore, none of 6, 8 or 9 exists.

To summarise; Γ_a has (314275), 7 exists, $7 = 7'$, and none of 6, 8, or 9 exists. To find new NSCs in this situation, we use paths that may lie outside disk D_a^2. By Lemma 2.4.4 (ii), every edge of L_a^2 belongs to a face , contained in D_a^2 that includes a vertex of L_a^1. Applying this to an edge of L_a^2 with at least one end in 1, we obtain a face g with at least one vertex v_1 of 1 and at least one vertex v_2 of 2b. The only components of $\Gamma_a \cap D_a^2$ that g may intersect are 1, 2 and (if $1 = 1'$) $1'$. By lemma 2.4.4 (ii), $g \cap 1$ and $g \cap 1'$ have at most one component each. Apply lemma 2.4.4 to g, letting E_1 and E_2 be the disks, with boundaries M_1 and M_2.

Let O_{12} be an arc from v_1 to v_2 inside g, and let O_{34} be an arc from an interior point v_3 of vw to a vertex v_4 of 4b inside $f_a \cap A_0$. Cut A_0 along O_{12} and O_{34}; the result is a disk with clockwise boundary (in compressed notation).

$$v_1 O_{12} v_2 \Gamma_a^{-1} v_4 O_{34}^{-1} v_3 \Gamma_a^{-1} v_2 O_{12}^{-1} v_1 \Gamma_a^{-1} v_3 O_{34} v_4 \Gamma_a^{-1} v_1$$

(We do not distinguish between two copies of $O_{12}, O_{34}, v_1, v_2, v_3, v_4$ since it will be clear which one we mean). This disk contains a slightly smaller disk R, whose boundary we divide into left, right, top and bottom segments for convenience. Reading bottom to top, R has $Q_1 = v_1\Gamma_a^{-1}3d(L_a^2)-14a\Gamma_a^{-1}v_1$ on the left and $Q_2 = v_2\Gamma_a 3aL_a^2 4c\Gamma_a v_2$ on the right. Reading left to right, R has O_{12} on the top and O_{12}^{-1} on the bottom. We use $>$, $<$, $\geq$, $\leq$ to denote order along Q_1 or Q_2, so that, for example, $u > v$ means u is above v. Along Q_1 we have $v_1 < 3d < 4a < v_1$, with $4a < 1' < v_1$ if $1' \neq 1$. Along Q_2 we have $v_2 < 2c < 7 < 5 < 3a < 4c < 2a < v_2$.

Now examining $(M_1 \cup M_2) \cap R$, there must be vertices $x_1 > x_2 > x_3 > x_4$ on Q_1, $y_1 > y_2 > y_3 > y_4$ on Q_2 and paths P_1, P_2, P_3, P_4 in R such that

(i) P_1, P_2, P_3, P_4 are vertex-disjoint, except that P_1 and P_2 may intersect at either both of v_1, v_2;

(ii) Each P_i has ends x_i, y_i and is otherwise disjoint from ∂R;

(iii) P_1 and P_4 are segments of M_1, while P_2 and P_3 are segments of M_2; and

(iv) $x_1 \in 1$, $y_1 \in 2$, $x_4 \in 1$ or $1'$, $y_4 \in 2$

The paths P_1 and P_4 are just the obvious segments of $\partial g = M_1$. If M_2 did not contain two disjoint paths P_2, P_3 as described, then we could find a circle in E_2 that was Non-contractible in S_3, contradicting the fact that E_2 is a disk.

If an end of P_2 or P_3 belongs to $(4aL_a^2 3d)^o$ or $(3aL_a^2 4c)^o$, then the path is not essential because it does not have both ends on Γ_a. However it can be extended to essential path in more than one way. Given $x_i > 3d$, define X_i^+ to be $4aL_a^2 x_i$ if $x_i < 4a$ or x_i if $x_i \geqslant 4a$. Given $x_i < 4a$, define X_i^- to be $x_i L_a^2 3d$ if $x_i > 3d$ or x_i otherwise. Define Y_i^+ and Y_i^- on the right similarly based on the relationship of y_i to $4c$ and $3a$.

Let w_5 be the last vertex of 5 along Γ_a. Note that $5L_a^2 4a = w_5 L_a^2 4a$. Suppose first that $x_3 < 4a$. Then necessarily $x_4 \in 1$. If $y_3 < w_5$ then $OP(P_1, X_3^- \cup P_3; 5L_a^2 4a, 7L_a^2 2a)$ produces a separating cycle which does not use 4b; replace vw by $vL_a^1 w$ to obtain a separating cycle avoiding vw. If $w_5 \leqslant y_3 < 4c$, then since $x_4 \in 1$ we have $AOP(X_3^- \cup P_3 \cup Y_3^-, P_4; 2cL_a^2 3a, 3dL_a^2 7 \cup 7 \cup 7L_a^2 2a)$. If $y_3 \geqslant 4c$ then we have the rather complicated $AOP(P_1, X_3^- \cup P_3 \cup y_3\Gamma_a^{-1}4c \cup 4c(L_a^2)^{-1}3a; 2cL_a^2 3a, 3dL_a^2 7 \cup 7\Gamma_a 5 \cup 5L_a^2 4a)$.

Now suppose that $x_3 \geqslant 4a$, and that $y_2 \leqslant 3a$. If $\Gamma_a \cap (3cL_a^1 2b)^o = \emptyset$ then we have $AOP(P_2, P_3; 2cL_a^2 3a, 3cL_a^2 2b)$. Otherwise 7 must intersect $3cL_a^1 2b$ so 7 has segments 7a, 7c, on L_a^2 and 7b on L_a^1. Then we have $AOP(3aL_a^2 4c, 4bL_a^1 3c; 3cL_a^1 7b, 3dL_a^2 7a)$.

Finally, suppose that $X_3 \geqslant 4a$ and $y_2 > 3a$. If $x_4 \in 1$, then $OP(P_2 \cup Y_2^+, P_4; 5L_a^2 4a, 7L_a^2 2a)$ produces a separating cycle that does not use 4b; replace vw by $vL_a^1 w$ to obtain a separating cycle avoiding vw. If $x_4 \notin 1$ then $1 \neq 1'$ and $x_4 \in 1'$. Let $P_2' = P_2 \cup y_2\Gamma_a^{-1}4c \cup 4c(L_a^2)^{-}3a$ if $y_2 \geqslant 4c$ and $P_2' = P_2 \cup Y_2^-$ if $3a < y_2 < 4c$. Then we have $AOP(P_4, P_2^-; 2cL_a^2 3a, 3dL_a^2 7 \cup 7\Gamma_a 5 \cup 5L_a^2 4a)$.

Since the embeddings Ψ_a and Ψ_b are equivalent by lemma 2.4.2, all operations applying on Γ_a also applies similarly on Γ_b. We have covered all cases, so this concludes the proof.

3 Conclusion.

After analysing Ellingman & Zhao's method and applying it to the proof of our main theorem it is found that the method works efficiently and our results are consistent with those obtained by them in their proof. Therefore it is concluded here that every 4-representative graph embedding on the triple torus contains two NSCs. The notions of equivalence in graph embeddings and that of homeomorphism of surfaces as defined topologically were used in extending Ellingman & Zhao's method to apply on a triple torus.

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